On a weak form of Ennola's conjecture about certain cubic number fields

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Motivation

Let $d \in \mathbb{Z}$ be square-free, and $K = \mathbb{Q}(\sqrt{d})$ (i.e., $[K : \mathbb{Q}] = 2$). Then the ring of integers of K (denoted $\mathcal{O}_K$) is

$$\mathcal{O}_K = \begin{cases} \{a + b\sqrt{d} | a, b \in \mathbb{Z}\}, & \text{if } d \equiv 2, 3 \pmod 4, \\ \{\frac{a}{2} + \frac{b}{2}\sqrt{d} | a, b \in \mathbb{Z}\}, & \text{if } d \equiv 1 \pmod 4. \end{cases}$$

For positive d, the group of units of K (denoted $\mathcal{O}_K^\times$) can be computed by using Pell's equations.

Motivation

Examples

- $K = \mathbb{Q}(\sqrt{5})$: $\mathcal{O}_K^\times = \langle -1, \frac{1+\sqrt{5}}{2} \rangle$.
- $K = \mathbb{Q}(\sqrt{11})$: $\mathcal{O}_K^\times = \langle -1, 10 + 3\sqrt{11} \rangle$.
- $K = \mathbb{Q}(\sqrt{15})$: $\mathcal{O}_K^\times = \langle -1, 4 + \sqrt{15} \rangle$.
- $K = \mathbb{Q}(\sqrt{17})$: $\mathcal{O}_K^\times = \langle -1, 4 + \sqrt{17} \rangle$.

Question

What happens if $K = \mathbb{Q}(\alpha)$ is a cubic number field?
(i.e., $[K : \mathbb{Q}] = 3$)
Namely, can we concretely determine $\mathcal{O}_K^\times$ for such K?

Motivation

Here, we concentrate on the case $K = \mathbb{Q}(\alpha)$: non-Galois totally real cubic number field.
Specifically, we are interested in the family of cubic number fields $K_l = \mathbb{Q}(\epsilon_l)$, where the minimal polynomial of ϵ_l is

$$X^3 + (l-1)X^2 - lX - 1 \in \mathbb{Z}[X] \tag{1}$$

with $l \in \mathbb{Z}_{\geq 3}$.

Motivation

By Dirichlet's unit theorem, the rank of $\mathcal{O}_{K_l}^{\times}$ is 2.
Since ϵ_l is a root of (1), ϵ_l itself is a unit in $\mathcal{O}_{K_l}^{\times}$.
From direct calculation, we know that $\epsilon_l - 1$ is also a unit.

Question

Is $\{\epsilon_l, \epsilon_l - 1\}$ a pair of fundamental units for $\mathcal{O}_{K_l}$? i.e.,
$\mathcal{O}_{K_l}^{\times} = \langle -1, \epsilon_l, \epsilon_l - 1 \rangle$ for $l \in \mathbb{Z}_{\geq 3}$?

Ennola's conjecture and its weak form

This is still an open problem.

Ennola's conjecture

For $l \in \mathbb{Z}_{\geq 3}$, $\mathcal{O}_{K_l}^{\times} = \langle -1, \epsilon_l, \epsilon_l - 1 \rangle$.

To reformulate Ennola's conjecture, we define

$$j_l = [\mathcal{O}_{K_l}^{\times} : \langle -1, \epsilon_l, \epsilon_l - 1 \rangle]$$

and call j_l *the unit index* of $\langle -1, \epsilon_l, \epsilon_l - 1 \rangle$ in $\mathcal{O}_{K_l}^{\times}$.
Now, Ennola's conjecture is equivalent to the following:

Ennola's conjecture

For $l \in \mathbb{Z}_{\geq 3}$, $j_l = 1$.

Ennola's conjecture and its weak form

There are several results on Ennola's conjecture.

Theorem (Ennola, 2004)

$\{\epsilon_l, \epsilon_l - 1\}$ *is a pair of fundamental units if* $[\mathcal{O}_{K_l} : \mathbb{Z}[\epsilon_l]] \leq l/3$.

Theorem (Louboutin, 2017)

$\gcd(j_l, 19!) = 1$ *for* $l \geq 3$, *and* $j_l = 1$ *for* $3 \leq l \leq 5 \cdot 10^7$.

Theorem (Louboutin, 2021)

Suppose abc*-conjecture holds. Then* $j_l = 1$ *for any sufficiently large* l. *i.e., Ennola's conjecture is true except for finitely many* l.

Ennola's conjecture and its weak form

Weak form of Ennola's conjecture (Louboutin, 2021)

Assume Conjectures 1 and 2, which will be introduced later, hold true. Then for any given prime $p \geq 3$, there are only finitely many $l \geq 3$ such that $p | j_l$.

Prof. Dohyeong Kim and I were able to prove that Conjectures 1 and 2 hold.

Weak form of Ennola's conjecture (Louboutin, Choi and Kim)

For any given prime $p \geq 3$, there are only finitely many $l \geq 3$ such that $p | j_l$.

Louboutin's Conjecture 1

Louboutin's conjectures are concerned with the family of following polynomials.

Definition

For $d \geq 1$, we define the the polynomial

$$P_d(X,Y) = d \sum_{\substack{k,l \geq 0 \\ 0 \leq 2k+3l \leq d}} (-1)^{k-1} \binom{k+l}{k} \binom{d-k-2l}{k+l} \frac{X^k Y^{d-2k-3l}}{d-k-2l} \in \mathbb{Z}[X,Y].$$

For small d's, $P_d(X,Y)$'s are easy to describe.

Louboutin's Conjecture 1

Examples

- $P_1(X,Y) = -Y$
- $P_2(X,Y) = -Y^2 + 2X$
- $P_3(X,Y) = -Y^3 + 3XY - 3$
- $P_4(X,Y) = -Y^4 + 4XY^2 - 2X^2 - 4Y$

Before we state Conjecture 1, we define

$$S_{a,b}(T) := \frac{1}{T^a} + \frac{1}{T^b} + T^{a+b},$$
$$R_{a,b}(T) := \frac{1}{T^a} + \frac{(-1)^{a+b}}{T^b} + T^{a+b},$$

where $a, b \in \mathbb{Z}$.

Louboutin's Conjecture 1

Conjecture 1

Let $a, b \in \mathbb{Z}$ be such that $a, b \neq 0$, and $c := a + b \neq 0$. Then for $d \in \{a, b, c\}$,

$$P_{|d|}(S_{a,b}(T), S_{a,b}(1/T)) = -S_{a,b}(1/T^{|d|}). \tag{2}$$

Moreover, if a is even and b is odd, then we have

$$P_{|d|}(-R_{a,b}(T), -R_{a,b}(1/T)) = \begin{cases} -S_{a,b}(1/T^{|d|}), & \text{if } d = a, \\ R_{a,b}(1/T^{|d|}), & \text{if } d \in \{b, c\}. \end{cases} \tag{3}$$

For instance, (2) is equivalent to

$$P_{|d|}\left(\frac{1}{T^a} + \frac{1}{T^b} + T^{a+b}, T^a + T^b + \frac{1}{T^{a+b}}\right) = -\left(T^{a|d|} + T^{b|d|} + \frac{1}{T^{(a+b)|d|}}\right).$$

Proof of Conjecture 1

The following proposition is a key to solve Conjecture 1.

Proposition

For any $d \geq 4$,

$$P_d(X,Y) = YP_{d-1}(X,Y) - XP_{d-2}(X,Y) + P_{d-3}(X,Y). \quad (4)$$

Proof.

Proceed by induction. Namely, we verify that the coefficients of $X^k Y^{d-2k-3l}$ in the LHS and the RHS of (4) coincide. □

Proof of Conjecture 1

The interpretation of (4) is as follows. Recall Newton identities of three variables.

Theorem (Newton identities)

Let $s_k = s_k(x_1, x_2, x_3) = x_1^k + x_2^k + x_3^k$ *be* k*-th power sum of three variables* x_1, x_2, x_3*. Define*
$\sigma_1 = x_1 + x_2 + x_3,\ \ \sigma_2 = x_1x_2 + x_2x_3 + x_3x_1,\ \ \sigma_3 = x_1x_2x_3$*. If* $d > 3$*, then*

$$s_d = \sigma_1 s_{d-1} - \sigma_2 s_{d-2} + \sigma_3 s_{d-3}. \tag{5}$$

Hence, $s_d = f_d(\sigma_1, \sigma_2, \sigma_3)$ *for some polynomial* f_d *in* $\sigma_1, \sigma_2, \sigma_3$ *for each* $d \geq 4$*.*

Proof of Conjecture 1

Examples

- $f_1(\sigma_1, \sigma_2, \sigma_3) = \sigma_1$
- $f_2(\sigma_1, \sigma_2, \sigma_3) = \sigma_1^2 - 2\sigma_2$
- $f_3(\sigma_1, \sigma_2, \sigma_3) = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3$
- $f_4(\sigma_1, \sigma_2, \sigma_3) = \sigma_1^4 - 4\sigma_1^2\sigma_2 + 2\sigma_2^2 + 4\sigma_1\sigma_3$.

Recall $P_d(X, Y)$'s in the previous section.

Examples

- $P_1(X, Y) = -Y$
- $P_2(X, Y) = -Y^2 + 2X$
- $P_3(X, Y) = -Y^3 + 3XY - 3$
- $P_4(X, Y) = -Y^4 + 4XY^2 - 2X^2 - 4Y$

Proof of Conjecture 1

In this setting, suppose $\sigma_3 = 1$.
Then σ_2 becomes

$$x_1x_2 + x_2x_3 + x_3x_1 = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3},$$

and (5) is

$$f_d(\sigma_1, \sigma_2, 1) = \sigma_1 f_{d-1}(\sigma_1, \sigma_2, 1) - \sigma_2 f_{d-2}(\sigma_1, \sigma_2, 1) + f_{d-3}(\sigma_1, \sigma_2, 1). \quad (6)$$

Proof of Conjecture 1

In terms of x_1, x_2, x_3, (6) is

$$\begin{aligned} x_1^d + x_2^d + x_3^d =& f_d\left(x_1 + x_2 + x_3, \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}, 1\right) \\ =&(x_1 + x_2 + x_3) f_{d-1}\left(x_1 + x_2 + x_3, \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}, 1\right) \\ &- \left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\right) f_{d-2}\left(x_1 + x_2 + x_3, \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}, 1\right) \\ &+ f_{d-2}\left(x_1 + x_2 + x_3, \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}, 1\right). \end{aligned}$$

Proof of Conjecture 1

Put $\sigma_2 = X$ and $\sigma_1 = Y$. Then (6) is

$$f_d(Y,X,1) = Y f_{d-1}(Y,X,1) - X f_{d-2}(Y,X,1) + f_{d-3}(Y,X,1). \qquad (7)$$

Compare (7) with (4):

$$P_d(X,Y) = Y P_{d-1}(X,Y) - X P_{d-2}(X,Y) + P_{d-3}(X,Y).$$

In fact, for any $d \geq 1$, $P_d(X,Y) = -f_d(Y,X,1)$.

Proof of Conjecture 1

Hence, we naturally obtain the following corollary:

Corollary

Let $x_1, x_2, x_3 \neq 0$ *be such that* $x_1 x_2 x_3 = 1$, *and* $d \geq 1$. *Then*

$$-\left(\frac{1}{x_1^d} + \frac{1}{x_2^d} + \frac{1}{x_3^d}\right) = P_d\left(x_1 + x_2 + x_3, \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\right).$$

Proof.

We know

$$P_d\left(x_1 + x_2 + x_3, \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\right) = -f_d\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}, x_1 + x_2 + x_3, 1\right),$$

and the f_d comes from Newton identities. □

Proof of Conjecture 1

Now we are ready to prove Conjecture 1.

Proof of Conjecture 1.

First, we verify (2). Put $x_1 = 1/T^a$, $x_2 = 1/T^b$, $x_3 = T^{a+b}$. Then $S_{a,b}(T) = x_1 + x_2 + x_3$, $x_1, x_2, x_3 \neq 0$, and $x_1 x_2 x_3 = 1$.
From the previous corollary,

$$
\begin{aligned}
-S_{a,b}(1/T^{|d|}) &= -\left(T^{a|d|} + T^{b|d|} + \frac{1}{T^{(a+b)|d|}}\right) = -\left(\frac{1}{x_1^d} + \frac{1}{x_2^d} + \frac{1}{x_3^d}\right) \\
&= P_d\left(x_1 + x_2 + x_3, \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\right) \\
&= P_{|d|}(S_{a,b}(T), S_{a,b}(1/T)).
\end{aligned}
$$

The proof of (3) directly follows from the same method, by replacing $x_1 \to -x_1$, $x_2 \to x_2$, and $x_3 \to -x_3$. □

Proof of Conjecture 1

Remark

One can directly use (4) to prove (by induction) for $d \geq 1$

$$P_d\left(\frac{1}{X}+\frac{1}{Y}+XY, X+Y+\frac{1}{XY}\right) = -\left(X^d+Y^d+\frac{1}{X^dY^d}\right),$$

which is equivalent to the previous corollary.

Remark

By the same argument as in the proof of the previous proposition, one can deduce

$$f_d(\sigma_1,\sigma_2,\sigma_3) = d \sum_{\substack{0\leq k,l \\ 0\leq 2k+3l\leq d}} (-1)^k \binom{k+l}{k}\binom{d-k-2l}{k+l}\frac{\sigma_1^{d-2k-3l}\sigma_2^k\sigma_3^l}{d-k-2l},$$

and if $\sigma_1 = x_1+x_2+x_3$, $\sigma_2 = x_1x_2+x_2x_3+x_3x_1$, $\sigma_3 = x_1x_2x_3$, then $f_d(\sigma_1,\sigma_2,\sigma_3) = x_1^d+x_2^d+x_3^d$.

Louboutin's Conjecture 2

We need more definitions to formalize Conjecture 2. For each case below, we define $0 \neq F_{a,b}(X,Y) = \sum_{u,v} f_{u,v} X^u Y^v$ as in the following table:

Cases	$F_{a,b}(X,Y)$
Case 1: $a \geq 1$ odd and $b \geq 1$ odd	$-P_a(Y,X) - P_b(Y,X) + P_c(X,Y)$
Case 2: $a \geq 1$ odd and $b \geq 1$ even	$-P_a(-Y,-X) - P_b(-Y,-X) + P_c(-X,-Y)$
Case 3: $a \geq 2$ even and $b \geq 1$ odd	$-P_a(-Y,-X) - P_b(-Y,-X) - P_c(-X,-Y)$

Table: $F_{a,b}(X,Y)$'s for each case.

Louboutin's Conjecture 2

Let $m \geq 3$ be an odd integer. With $a, b \in \mathbb{Z}$, we define

$$R_{a,b,m}(T) = R_{a,b}(T) + \frac{b-a}{m}T^{-a-m} + \frac{(-1)^{a+b}(a-2b)}{m}T^{-b-m} + \frac{b}{m}T^{a+b-m},$$
$$G_{a,b,m}(T) = F_{a,b}(R_{a,b,m}(T), R_{-a,-b,m}(T)),$$
$$N_{a,b,m} = -\deg G_{a,b,m}(T),$$
$$s = \deg R_{a,b}(T), \quad t = \deg R_{-a,-b}(T),$$
$$M_{a,b} = \max\{us + vt : f_{u,v} \neq 0\}.$$

Conjecture 2

Assume that $(a, b) \in \mathbb{Z}_{\neq 0} \times \mathbb{Z}_{\geq 1}$ is not of the form $(-2b, b)$ with $b \geq 1$ odd, (b, b) with $b \geq 1$ odd, $(-b/2, b)$ with $b \geq 2$ even. Let $m_{a,b} = a^2 + ab + b^2 \geq 5$ be odd. Then $M_{a,b} = (a + b)\max(a, b)$, $N_{a,b,m_{a,b}} = \min(a, b)^2$ for any cases listed in the previous table.

Proof of Conjecture 2

For each case in the table, we define

$$G_{a,b}(T) := F_{a,b}(R_{a,b}(T), R_{a,b}(1/T)).$$

Recall

$$R_{a,b}(T) = \frac{1}{T^a} + \frac{(-1)^{a+b}}{T^b} + T^{a+b}.$$

Louboutin showed that Conjecture 1 implies the following lemma.

Proof of Conjecture 2

Lemma

$G_{a,b}(T)$'s for each case are given as the following table:

Cases	$G_{a,b}(T)$
Case 1	$T^{-a^2} + T^{-b^2} - T^{-c^2} + 2T^{-ab}$
Case 2	$-T^{-a^2} + T^{-b^2} + T^{-c^2} + 2T^{-ab}$
Case 3	$T^{-a^2} + T^{-b^2} - T^{-c^2}$

Table: $G_{a,b}(T)$'s in each case.

Proof of Conjecture 2

We set

$$E_{a,b}(T) = \frac{b-a}{T^a} + \frac{(-1)^{a+b}(a-2b)}{T^b} + bT^{a+b}.$$

For notational convenience, we let $m_{a,b} = m$ from now on. Then

$$\begin{aligned} R_{a,b,m}(T) &= R_{a,b}(T) + \frac{1}{mT^m} E_{a,b}(T), \\ R_{-a,-b,m}(T) &= R_{-a,-b}(T) + \frac{1}{mT^m} E_{-a,-b}(T) \\ &= R_{a,b}\left(\frac{1}{T}\right) - \frac{1}{mT^m} E_{a,b}\left(\frac{1}{T}\right). \end{aligned}$$

Proof of Conjecture 2

We only sketch the proof of Conjecture 2. For Case 1,

$$F_{a,b}(X,Y) = -P_a(Y,X) - P_b(Y,X) + P_c(X,Y)$$

and

$$\begin{aligned} G_{a,b,m}(T) =& F_{a,b}\left(R_{a,b}(T) + \frac{1}{mT^m}E_{a,b}(T), R_{-a,-b}(T) + \frac{1}{mT^m}E_{-a,-b}(T)\right) \\ =& -P_a\left(R_{a,b}\left(\frac{1}{T}\right) - \frac{1}{mT^m}E_{a,b}\left(\frac{1}{T}\right), R_{a,b}(T) + \frac{1}{mT^m}E_{a,b}(T)\right) \\ & - P_b\left(R_{a,b}\left(\frac{1}{T}\right) - \frac{1}{mT^m}E_{a,b}\left(\frac{1}{T}\right), R_{a,b}(T) + \frac{1}{mT^m}E_{a,b}(T)\right) \\ & + P_c\left(R_{a,b}(T) + \frac{1}{mT^m}E_{a,b}(T), R_{a,b}\left(\frac{1}{T}\right) - \frac{1}{mT^m}E_{a,b}\left(\frac{1}{T}\right)\right). \end{aligned}$$

Proof of Conjecture 2

Now we explicitly describe

$$-P_a\left(R_{a,b}\left(\frac{1}{T}\right)-\frac{1}{mT^m}E_{a,b}\left(\frac{1}{T}\right), R_{a,b}(T)+\frac{1}{mT^m}E_{a,b}(T)\right)$$

using the formula

$$P_d(X,Y)=d\sum_{\substack{k,l\geq 0\\ 0\leq 2k+3l\leq d}}(-1)^{k-1}\binom{k+l}{k}\binom{d-k-2l}{k+l}\frac{X^kY^{d-2k-3l}}{d-k-2l}.$$

Proof of Conjecture 2

$$
\begin{aligned}
&-P_a\left(R_{a,b}\left(\frac{1}{T}\right)-\frac{1}{mT^m}E_{a,b}\left(\frac{1}{T}\right), R_{a,b}(T)+\frac{1}{mT^m}E_{a,b}(T)\right)\\
=&a\sum_{\substack{k,l\geq 0\\ 0\leq 2k+3l\leq a}}(-1)^k\binom{k+l}{k}\binom{a-k-2l}{k+l}\frac{1}{a-k-2l}\cdot\\
&\left(R_{a,b}\left(\frac{1}{T}\right)-\frac{1}{mT^m}E_{a,b}\left(\frac{1}{T}\right)\right)^k\left(R_{a,b}(T)+\frac{1}{mT^m}E_{a,b}(T)\right)^{a-2k-3l}\\
=&-P_a\left(R_{a,b}\left(\frac{1}{T}\right), R_{a,b}(T)\right)\\
&\underbrace{+a\sum_{\substack{k,l\geq 0\\ 0\leq 2k+3l\leq a}}(-1)^k\binom{k+l}{k}\binom{a-k-2l}{k+l}\frac{1}{a-k-2l}\sum_{\substack{0\leq i\leq k\\ 0\leq j\leq a-2k-3l\\ (i,j)\neq(0,0)}}A_{k,i}^{a-2k-3l,j}}_{=A_1(T)},
\end{aligned}
$$

Proof of Conjecture 2

where

$$A_{k,i}^{a-2k-3l,j} = \binom{k}{i} \left(\frac{-1}{mT^m} E_{a,b}\left(\frac{1}{T}\right)\right)^i \left(R_{a,b}\left(\frac{1}{T}\right)\right)^{k-i} \cdot \binom{a-2k-3l}{j} \left(\frac{1}{mT^m} E_{a,b}(T)\right)^j (R_{a,b}(T))^{a-2k-3l-j}.$$

One can easily verify that

$$\deg\left(A_1(T)\right) = -b^2.$$

Proof of Conjecture 2

We repeat the similar process to $-P_b(Y, X)$ and $P_c(X, Y)$ and know that

$$
\begin{aligned}
& -P_b\left(R_{a,b}\left(\frac{1}{T}\right) - \frac{1}{mT^m}E_{a,b}\left(\frac{1}{T}\right), R_{a,b}(T) + \frac{1}{mT^m}E_{a,b}(T)\right) \\
= & -P_b\left(R_{a,b}\left(\frac{1}{T}\right), R_{a,b}(T)\right) + (\text{degree } -a^2 \text{ polynomial } B_1(T)), \\
& P_c\left(R_{a,b}(T) + \frac{1}{mT^m}E_{a,b}(T), R_{a,b}\left(\frac{1}{T}\right) - \frac{1}{mT^m}E_{a,b}\left(\frac{1}{T}\right)\right) \\
= & P_c\left(R_{a,b}(T), R_{a,b}\left(\frac{1}{T}\right)\right) + (\text{degree } -\min(a,b)^2 \text{ polynomial } C_1(T)).
\end{aligned}
$$

Proof of Conjecture 2

Hence, we know (by the previous lemma)

$$\begin{aligned} G_{a,b,m}(T) =& G_{a,b}(T) + (\text{degree } -\min(a,b)^2 \text{ polynomial}) \\ =& T^{-a^2} + T^{-b^2} - T^{-c^2} + 2T^{-ab} \\ & + (\text{degree } -\min(a,b)^2 \text{ polynomial}). \end{aligned}$$

We can also observe that the coefficient of $T^{-\min(a,b)^2}$ in $G_{a,b,m}(T)$ is nonzero. Hence, $N_{a,b,m} = \min(a,b)^2$ for Case 1. We can show $N_{a,b,m} = \min(a,b)^2$ for other cases in the same way.

Proof of Conjecture 2

$M_{a,b} = (a+b)\max(a,b)$ can be checked in a similar way.
By definitions of $M_{a,b}$ and $F_{a,b}$ in Table 1, $M_{a,b}$ should be one of the followings $(c = a+b)$:

$$\max\{(a+b)(a-2k-3l) + k\max(a,b) : 0 \le k,l,\ 0 \le 2k+3l \le a\}, \quad (8)$$

$$\max\{(a+b)(b-2k-3l) + k\max(a,b) : 0 \le k,l,\ 0 \le 2k+3l \le b\}, \quad (9)$$

$$\max\{(a+b)k + (c-2k-3l)\max(a,b) : 0 \le k,l,\ 0 \le 2k+3l \le c\}. \quad (10)$$

Regardless of $\max(a,b)$, (8), (9), (10) attain maxima at $k = l = 0$.
If $\max(a,b) = a$, then (8) $= a(a+b)$, (9) $= b(a+b)$, (10) $= a(a+b)$ and clearly $M_{a,b} = a(a+b) = (a+b)\max(a,b)$.
We can show $M_{a,b} = a(a+b) = (a+b)\max(a,b)$ when $\max(a,b) = b$ in the same way.

Thank you for your attention!