3. Combinatorial realization of crystals.

- Tableaux realization of $B(\lambda)$ for $U_q(\mathfrak{gl}_n)$
- Decomposition of $U_q(\mathfrak{gl}_n)$-modules
- Applications to the theory of symmetric functions.

  : LR coeff. RSK correspondence.

Tableaux realization

Suppose $\mathfrak{g} = \mathfrak{gl}_n$

Recall that we have constructed a crystal base of $V(\ell\varpi_1)$, $V(\varpi_k)$

where the crystal can be identified as a set

$$B(\ell\varpi_1) = \left\{ \begin{array}{|c|c|c|c|}\hline a_1 & a_2 & \cdots & a_\ell \\ \hline\end{array} \;\middle|\; 1 \leq a_1 \leq a_2 \leq \cdots \leq a_\ell \leq n \right\}$$

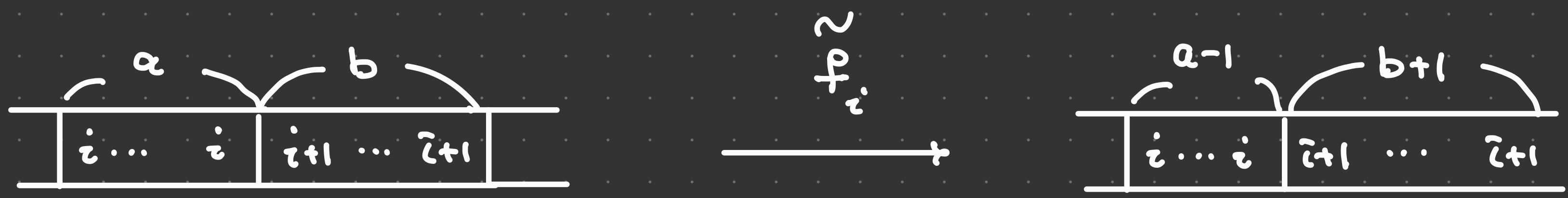
$$B(\varpi_k) = \left\{ \begin{array}{|c|}\hline a_1 \\ \hline \vdots \\ \hline a_k \\ \hline\end{array} \;\middle|\; 1 \leq a_1 < \cdots < a_k \leq n \right\}$$

$B = B(\ell\varpi_1)$ or $B(\varpi_k)$

$T \in B \qquad \mathrm{wt}(T) = \sum_{i \geq 1} \delta_{a_i} = \delta_{a_1} + \delta_{a_2} + \cdots$

$$\tilde{f}_i T = \begin{cases} T' \in B & \text{obtained from } T \text{ by replacing } i \text{ w/ } i+1 \\ 0 & \end{cases}$$

e.g.



- One can realize $B(\lambda)$ for $\lambda \in P_+$ using B as building block.

  (like fundamental representations in repn's of semisimple Lie alg's)

- The basic strategy is to describe the connected component

$$C(b) \subset B(\varpi_{k_1}) \otimes \cdots \otimes B(\varpi_{k_r}) \quad \text{or} \quad B(\ell_1\varpi_1) \otimes \cdots \otimes B(\ell_s\varpi_1)$$

where $\tilde{e}_i b = 0$ for all i $\quad \mathrm{wt}(b) = \lambda$. $\quad \big(\Rightarrow C(b) \cong B(\lambda) \big)$

* This can be applied to any of

In particular,

$$B(\varpi_k) \subset B(\varpi_1)^{\otimes k} \qquad\qquad B(\ell\varpi_1) \subset B(\varpi_1)^{\otimes \ell}$$

$$\begin{array}{|c|}\hline a_1 \\ \hline \vdots \\ \hline a_k \\ \hline \end{array} \longmapsto \boxed{a_1} \otimes \cdots \otimes \boxed{a_k} \qquad\qquad \begin{array}{|c|c|c|}\hline a_1 & \cdots & a_\ell \\ \hline \end{array} \longmapsto \boxed{a_\ell} \otimes \cdots \otimes \boxed{a_1}$$

$\lambda \in P_+ \qquad \lambda = \sum_{a=1}^{n} \lambda_a \delta_a \qquad (\lambda_1 \geq \cdots \geq \lambda_n) \in \mathbb{Z}^n$

λ : polynomial if $\lambda_i \geq 0 \ \forall i$.

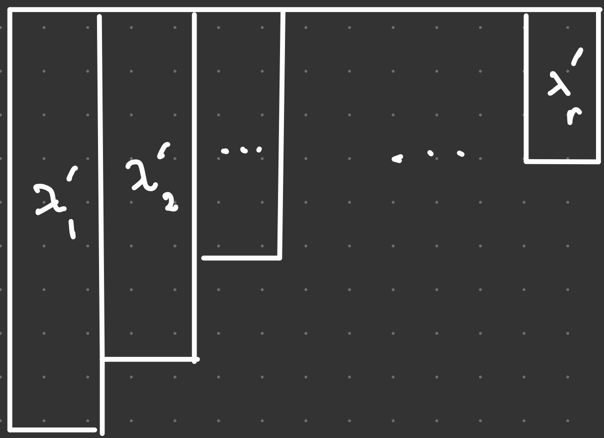
λ : polynomial $\longleftrightarrow$ $\lambda = (\lambda_1, \cdots, \lambda_n)$: partition

$SST_n(\lambda)$ = the set of semi-standard tableaux of shape λ with the entries in $\{1, \cdots, n\}$

e.g.

$B(\varpi_k) = SST_n(1^k) \qquad B(\ell\varpi_1) = SST_n(\ell)$

λ : a partition $\qquad \mu = \lambda'$: the conjugate of λ

$\lambda =$ 

$$\begin{array}{l} \lambda'_j \\ \parallel \\ \mu_j \end{array}$$: the length of j th column.

Want to describe $B(\lambda)$ in

$$B(\varpi_{\mu_r}) \otimes \cdots \otimes B(\varpi_{\mu_1}) = SST_n(1^{\mu_r}) \otimes \cdots \otimes SST_n(1^{\mu_1})$$

$$\cup \quad \text{as a set}$$

$$SST_n(\lambda)$$

regarding j th column

as an elt in $SST_n(1^{\mu_j})$

Highest weight vectors :

$$B(\varpi_{\mu_r}) \otimes \cdots \otimes B(\varpi_{\mu_1})$$

$$\Cup$$

$v_{\omega_\mu} = v_{\varpi_{\mu_r}} \otimes \cdots \otimes v_{\varpi_{\mu_1}}$ the cnn. comp of $v_{\varpi_\mu} \cong B(\lambda)$

$$SST_n(1^{\mu_r}) \otimes \cdots \otimes SST_n(1^{\mu_1})$$

$$v_{\omega_\mu} = v_{\varpi_{\mu_r}} \otimes \cdots \otimes v_{\varpi_{\mu_1}} \longmapsto H_{(1^{\mu_r})} \otimes \cdots \otimes H_{(1^{\mu_1})} = H_\lambda$$

$$H_{(1^\ell)} = \boxed{\begin{array}{c} 1 \\ 2 \\ \vdots \\ \ell \end{array}}$$

So. it is enough to consider cnn. comp. of H_λ. $\subset B(\varpi_1)^{\otimes |\lambda|}$

The following formula is very useful ("signature rule")

B_1, B_2 : crystals . $b_1 \otimes b_2 \in B_1 \otimes B_2$ $\quad i \in I$

$$\sigma_i = (\underbrace{- \cdots -}_{\varepsilon_i(b_1)} \underbrace{+ \cdots +}_{\varphi_i(b_1)}) \cdot (\underbrace{- \cdots -}_{\varepsilon_i(b_2)} \underbrace{+ \cdots +}_{\varphi_i(b_2)})$$

i) replace any two neighboring $(+,-)$ w/ $(\cdot, \cdot)$

ii) repeat the process i) (ignoring $\cdot$) as far as possible

to have a seq. of the form

$$\overline{\sigma_i} = (- \cdots - + \cdots +) \quad (\text{ignoring } \cdot)$$

Example

Then

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} (\tilde{e}_i b_1) \otimes b_2 & \text{if } \exists - \text{ in } \overline{\sigma_i} \in b_1 \\ b_1 \otimes (\tilde{e}_i b_2) & \text{otherwise.} \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} (\tilde{f}_i b_1) \otimes b_2 & \text{if } \exists + \text{ in } \overline{\sigma_i} \in b_1 \\ b_1 \otimes (\tilde{f}_i b_2) & \text{otherwise.} \end{cases}$$

The above combinatorial rule can be applied to

$$B_1 \otimes \cdots \otimes B_r \quad (r \geq 2)$$

Example $B = V(\varpi_1) = SST_n(1)$

$$B^{\otimes k} \ni b_1 \otimes \cdots \otimes b_k = b \quad (b_i \in SST_n(1))$$

$$\|$$

$b_1 \cdots b_k$: a word of length k w/ letters in $\{1, \dots, n\}$

$$\begin{array}{cccccc} 1 & 2 & 1 & 3 & 3 & 1 \\ + & - & + & \cdot & \cdot & + \\ \cancel{+} & \cancel{-} & + & \cdot & \cdot & + \end{array} \xrightarrow{\tilde{f}_1} \begin{array}{cccccc} 1 & 2 & 2 & 3 & 3 & 1 \\ \cancel{+} & \cancel{-} & - & \cdot & \cdot & + \end{array} \xrightarrow{\tilde{f}_1} 1\ 2\ 2\ 3\ 3\ 2.$$

$$\begin{array}{cccccc} 1 & 2 & 1 & 3 & 3 & 1 \\ \cdot & + & \cdot & - & - & \cdot \\ \cdot & \cancel{+} & \cdot & \cancel{-} & - & \cdot \end{array} \xleftarrow{\tilde{f}_2} 1\ 2\ 1\ 3\ 2\ 1$$

Then

① $H_\lambda \in SST_n(\lambda)$

② $SST_n(\lambda) \cup \{0\}$ closed under $\tilde{e}_i, \tilde{f}_i$ $(i \in I)$

③ $SST_n(\lambda)$ is connected as an I-colored oriented graph.

(i.e. any $T \in SST_n(\lambda)$ is connected to H_λ)

$\therefore$ $SST_n(\lambda) \cong B(\lambda)$

We may obtain the same result by considering

$$B(\lambda) \subset B(\lambda_1 \varpi_1) \otimes \cdots \otimes B(\lambda_n \varpi_1) = SST_n(\lambda_1) \otimes \cdots \otimes SST_n(\lambda_n)$$

Decomposition of $V^{\otimes k}$ (V : natural repn of $U_q(\mathfrak{gl}_n)$)

The following lemma is also useful.

Lem $B_1, \cdots, B_r$: crystals

$b = b_1 \otimes \cdots \otimes b_r \in B_1 \otimes \cdots \otimes B_r$: maximal (i.e. $\forall i \ \tilde{e}_i b = 0$)

$\iff b_1 \otimes \cdots \otimes b_i$: maximal for all $1 \leq i \leq r$

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pf.) Use induction.

$r = 2$

$b_1 \otimes b_2$: maximal $\Rightarrow$ b_1 : maximal (by tensor product rule.) ☒

$(\mathcal{L}, B)$: the crystal base of V

<u>Lem</u> $B^{\otimes k} \ni b_1 \otimes \cdots \otimes b_k = b$

b : maximal $\Longleftrightarrow$ b : a lattice word i.e.

$\forall_{1 \le i \le k}$, $a \in \{1, \cdots, n\}$

$\#$ of a in $b_1 \cdots b_i$ $\geq$ $\#$ of $a+1$ in $b_1 \cdots b_i$

<u>Ex</u>. $1\,1\,2\,3\,1\,2\,3$ (o) $1\,2\,2\,3$ (x)

pf.) b : a lattice word $\Longleftrightarrow$ $\underbrace{i\text{-signature of } b}_{\overline{\sigma_i}}$ has no "$-$"

$\boxtimes$

$(B^{\otimes k})^{h.w} =$ the set of maximal vectors in $B^{\otimes k}$

$b_1 \cdots b_k = b_1 \otimes \cdots \otimes b_k \in (B^{\otimes k})^{h.w.}$: lattice word

$\lambda^{(i)} = wt(b_1 \cdots b_i)$: a partition or Young diagram $(1 \leq i \leq k)$

$$\emptyset \subset \lambda^{(1)} \subset \lambda^{(2)} \subset \cdots \subset \lambda^{(k)} = \lambda \qquad *$$

where $\lambda^{(i-1)} \longrightarrow \lambda^{(i)}$ adding a box at b_ith row.

Ex. 1 1 2 3 1 2 3

1	2	5
3	6	
4	7	

$$\mathcal{C}(b_1 \otimes \cdots \otimes b_k) \cong B(\lambda)$$

# of cnn. components in $B^{\otimes k} \cong B(\lambda)$

= # of seq of partitions $\lambda^{(1)} \subset \cdots \subset \lambda^{(k)} = \lambda$ in (*)

= # of standard Young tableaux of shape λ (= dim Specht module S^λ)

=: f_λ

$$B^{\otimes k} \cong \coprod_{\lambda \vdash k} B(\lambda)^{f_\lambda}$$

This implies $V^{\otimes k} \cong \bigoplus_{\lambda \vdash k} V(\lambda)^{\oplus f_\lambda}$

Rmk ① $\exists$ a q-analogue Schur-Weyl duality

$$V^{\otimes k} = \bigoplus_{\substack{\ell(\lambda)\leq n \\ \lambda \vdash k}} V(\lambda) \otimes S^{\lambda} \quad \circlearrowleft \left(U_q(\mathfrak{gl}_n), \mathcal{H}_q(S_n) \right)$$

② $\exists$ an explicit isomorphism of $\mathfrak{gl}_n$-crystals.

$$B^{\otimes k} \longrightarrow \coprod_{\substack{\lambda \vdash k \\ \ell(\lambda)\leq n}} SST_n(\lambda) \times ST_k(\lambda)$$

$$w = b_1 \cdots b_k \longmapsto (P(w), Q(w))$$

constant on each cnn. comp.!

Robinson-Schensted correspondence

insertion tableau

recording tableau

(cf. Fulton. "Young tableaux")

Littlewood-Richardson rule

Recall

$$B(\mu) \otimes B(\nu) = \coprod_{\lambda \in P_+} B(\lambda)^{\oplus c_{\mu\nu}^{\lambda}}$$

$$c_{\mu\nu}^{\lambda} = \# \underbrace{\left\{ b \in B(\nu) \;\middle|\; \begin{array}{l} \varepsilon_i(b) \leq \langle h_i, \mu \rangle \quad (i \in I) \\ \mu + \mathrm{wt}(b) = \lambda \end{array} \right\}}_{B(\nu)^{\circ}}$$

$$b_2 \in B(\nu)^{\circ} \iff \begin{array}{l} \tilde{e}_i(b_1 \otimes b_2) = 0 \text{ for all } i \quad (\Rightarrow b_1 = u_{\mu}) \\ \mathrm{wt}(b_1 \otimes b_2) = \lambda \end{array}$$

Characterization of $B(\nu)^\circ$

Assume

$$B(\nu) = SST_n(\nu) \subset SST_n(1^{\nu'_r}) \otimes \cdots \otimes SST_n(1^{\nu'_1}) \subset SST_n(1)^{\otimes |\nu|}$$

$b \in B(\nu)^\circ$

$b = T_r \otimes \cdots \otimes T_1$ (T_i : the ith column from the right)

$$= \underbrace{\left(w_1^{(r)} \cdots w_{\nu'_r}^{(r)} \right)}_{\text{col word of } T_r} \cdots \underbrace{\left(w_1^{(1)} \cdots w_{\nu'_1}^{(1)} \right)}_{\text{col. word of } T_1} = w_1 \cdots w_{|\nu|}$$

$H_\mu \otimes w_1 \otimes \cdots \otimes w_k$: maximal for all $1 \le k \le |\nu|$

$\lambda^{(k)} = \mathrm{wt}\left(H_\mu \otimes w_1 \otimes \cdots \otimes w_k \right)$: a Young diagram

We have a seq of Young diagrams

$$\mu = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(|\nu|)} =: \lambda \qquad (*)$$

Conversely. for $b \in B(\nu)$,

$\lambda^{(k)} = \mathrm{wt}\left(H_\mu \otimes w_1 \otimes \cdots \otimes w_k\right)$: a partition for all k

$\Rightarrow b \in B(\nu)^\circ$

$$\therefore\ B(\nu)^\circ = \left\{ b = b_1 \otimes \cdots \otimes b_{|\nu|} \;\middle|\; \begin{array}{l} \lambda^{(k)} = \mathrm{wt}\left(H_\mu \otimes b_1 \otimes \cdots \otimes b_k\right) \\ \text{: a partition for all } k \end{array} \right\}$$

$$c^{\lambda}_{\mu\nu} = \# \left\{ b = b_1 \otimes \cdots \otimes b_{|\nu|} \;\middle|\; \begin{array}{l} \lambda^{(k)} = \mathrm{wt}(H_\mu \otimes b_1 \otimes \cdots \otimes b_k) \\ \quad : \text{a partition for all } k \\ \lambda^{(|\nu|)} = \lambda \end{array} \right\}$$

<u>Rmk</u> ① $b \in B(\nu)^\circ$

S : a tableau of shape λ/μ where $\lambda^{(k)}/\lambda^{(k-1)}$ is filled with j if $b_k = \underline{w_j^{(i)}}$

entry in the j^{th} row of T.

We have

$$\begin{array}{ccc} B(\nu)^\circ & \xrightarrow{1-1} & \underline{LR^{\lambda}_{\mu\nu}} \\ b & \longmapsto & S \end{array}$$

the set of Littlewood-Richardson tableaux of shape λ/μ with content ν.

Example

$$\mu = \begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \end{array} \qquad \nu = \begin{array}{cc} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \end{array}$$

$$T = \begin{array}{cc} 1 & 2 \\ 2 & 3 \\ 3 & \end{array} = 2 \otimes 3 \otimes 1 \otimes 2 \otimes 3 = \begin{array}{ccccc} 2_1 & 3_2 & 1_1 & 2_2 & 3_3 \\ 1 & 2 & 3 & 4 & 5 \end{array} = b_1 \cdots b_5$$

index of rows in μ

index in $\otimes$

$$H_\mu = \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & \end{array}$$

$\mathrm{wt}(H_\mu \otimes b_1 \cdots b_k)$ for $1 \leq k \leq 5$ LR tableau of sh λ/μ conf ν

$$\begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & 3 \\ \bullet & \bullet & \bullet & 1 & 4 \\ 2 & 5 & & & \end{array} \longrightarrow \begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & 1 \\ \bullet & \bullet & \bullet & 1 & 2 \\ 2 & 3 & & & \end{array}$$

② $\exists$ analogues of realization of $B(\lambda)$ for B_n, C_n, D_n

(called Kashiwara-Nakashima tableaux)

$\exists$ analogues of the formula for $c^{\lambda}_{\mu\nu}$ in case of B_n, C_n, D_n

using KN tableaux. (Nakashima)

③ In general, once we have a realization of $B(\lambda)$

($\mathfrak{g}$: not necessarily of finite type) , then we have

a formula of $c^{\lambda}_{\mu\nu}$. or $B(\nu)^{\circ}$. depending on its model.

Howe duality of type A & crystals

$S(\mathbb{C}^m \otimes \mathbb{C}^n)$: a repn. of (GL_m, GL_n) and hence $(\mathfrak{gl}_m, \mathfrak{gl}_n)$

$$S(\mathbb{C}^m \otimes \mathbb{C}^n) \cong \bigoplus_{\substack{\lambda : \text{par.} \\ \ell(\lambda) \leq m, n}} V_m(\lambda) \otimes V_n(\lambda) \qquad (*)$$

The above decomposition can be obtained by the theory of reductive dual pairs by Howe

The char. of $(*)$ is the well-known identity.

$$\frac{1}{\prod_{i,j}(1 - x_i y_j)} = \sum_{\lambda} \underline{s_\lambda(x)}\, s_\lambda(y) \qquad \text{Cauchy identity.}$$

Schur polynomial.

Similarly, we have

$$\Lambda(\mathbb{C}^m \otimes \mathbb{C}^n) \cong \bigoplus_{\substack{\ell(\lambda)\leq m \\ \ell(\lambda')\leq n}} V_m(\lambda) \otimes V_n(\lambda') \qquad (*')$$

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q-analogues

To have q-analogues of $(*)$ and $(*')$, we need q-deform of

$S(\mathbb{C}^m \otimes \mathbb{C}^n)$ and $\Lambda(\mathbb{C}^m \otimes \mathbb{C}^n)$ with an action of $(U_q(gl_m), U_q(gl_n))$

$S(\mathbb{C}^m \otimes \mathbb{C}^n) \rightsquigarrow$ a quantum co-ordinate ring $k_q[M_{m,n}]$ $\qquad S_q(m,n)$

$\Lambda(\mathbb{C}^m \otimes \mathbb{C}^n) \rightsquigarrow$ a skew analogue of $k_q[M_{m,n}]$. $\qquad \Lambda_q(m,n)$

$S_q(m,n)$ = the assoc. K-alg. gen. by X_{ij} $1\le i\le m,\ 1\le j\le n$.

subject to the following relations

$$X_{jk}X_{ik} = q X_{ik}X_{jk} \qquad \text{if } j>i$$

$$X_{il}X_{ik} = q X_{ik}X_{il} \qquad \text{if } l>k$$

$$X_{ij}X_{kl} = X_{kl}X_{ij} \qquad \text{if } i<k,\ j>l$$

$$X_{ij}X_{kl} - X_{kl}X_{ij} = (q-q^{-1})X_{ik}X_{jl} \qquad \text{if } i<k,\ j<l.$$

<u>Rmk</u> $\underline{S_q(m,n) \cong U_q^-(w_{(m,n)})}$ $w_{(m,n)}$: the longest elt. in $S_{m+n}/S_m\times S_n$

$$X_{ij} \longleftrightarrow f_\beta \qquad (\beta : \text{not a root of } \mathfrak{gl}_m\oplus\mathfrak{gl}_n)$$

Relations of X_{ij} $\longleftrightarrow$ Levendorskii-Soibelman relations

$\exists$ an action of $U_q(\mathfrak{gl}_m)$ on $S_q(m,n)$

$$e_i X_{ab} = \delta_{a+1\,i} X_{ib} \quad f_i X_{ab} = \delta_{ai} X_{i+1\,b} \quad \mathrm{wt}(X_{ab}) = \delta_a .$$

$$t_i(XY) = t_i(X)\, t_i(Y)$$

$$e_i(XY) = e_i(X)\, t_i^{-1}(Y) + X\, e_i(Y)$$

$$f_i(XY) = f_i(X)\, Y + t_i(X)\, f_i(Y)$$

Similarly $\exists$ $U_q(\mathfrak{gl}_n)$ - action on $S_q(m,n)$ commuting w/ $U_q(\mathfrak{gl}_m)$

$$\overset{\shortparallel}{\langle e_i^*, f_i^*, t_i^{*\,\pm 1} \rangle}$$

$$x_i^* X_{ab} := (x_i X_{ba})^t \qquad \left(X_{ab}^t = X_{ba} \right)$$

Rmk

$S_q^{(j)}$ = the subalg. gen. by X_{ij} $(1 \leq i \leq m)$ $\cong \bigoplus_{l \geq 0} V_m(l\varpi_1)$

$S_q \cong S_q^{(1)} \otimes \cdots \otimes S_q^{(n)}$ as a $U_q(gl_m)$-module.

$M = (m_{ij}) \in M_{m.n}(\mathbb{Z}_{\geq 0})$

$$X^M := \overrightarrow{\prod_{i,j}} \frac{X_{ij}^{m_{ij}}}{[m_{ij}]!}$$ where $\overrightarrow{\prod}$: lexico-graphic

$\mathcal{L} = \bigoplus A_0 X^M$ $\quad$ $\mathcal{B} = \{ x^M \ (\mathrm{mod}\ q\mathcal{L}) \}$

$(\mathcal{L}, \mathcal{B})$: a crystal base of $S_q(m.n)$ over $U_q(gl_m) \oplus U_q(gl_n)$

We may identify $\mathcal{B}$ with $M_{m,n}(\mathbb{Z}_{\geq 0}) = M_{m,n}$

$$M_{m,n} \cong \coprod_{(\ell_1,\cdots,\ell_n)\in\mathbb{Z}_+^n} B_m(\ell_1\varpi_1)\otimes\cdots\otimes B_m(\ell_n\varpi_1) \quad \text{as a } U_q(\mathfrak{gl}_m)\text{-crystal}$$

$$M \longmapsto M^1 \otimes \cdots \otimes M^n$$

M^j : the j^{th} col. $\in B_m(\ell_j\varpi_1)$ w/ $\ell_j = \sum_i m_{ij}$

$$\cong \coprod_{(\ell'_1,\cdots,\ell'_m)\in\mathbb{Z}_+^m} B_n(\ell'_1\varpi_1)\otimes\cdots\otimes B_n(\ell'_m\varpi_1) \quad \text{as a } U_q(\mathfrak{gl}_n)\text{-crystal}$$

$$M \longmapsto M_1 \otimes \cdots \otimes M_m$$

M_i : the i^{th} row $\in B_n(\ell'_i\varpi_1)$ w/ $\ell'_i = \sum_j m_{ij}$

$\tilde{e}_i, \tilde{f}_i$: crystal operators for $U_q(\mathfrak{gl}_m)$

$\tilde{e}_j^*, \tilde{f}_j^*$: crystal operators for $U_q(\mathfrak{gl}_n)$

We have $\qquad \underline{\tilde{x}_i \tilde{y}_j^* = \tilde{y}_j^* \tilde{x}_i} \qquad (x, y \in \{e, f\})$

$$M_{m,n}^{h.w} := \{ M \mid \tilde{e}_i M = \tilde{e}_j^* M = 0 \text{ for all } i, j \}$$

$$= \left\{ M = (m_{ij}) \;\middle|\; m_{ij} = 0 \ (i \neq j),\ m_{11} \geq m_{22} \geq \cdots \right\}$$

$$C(M) \cong B_m(\lambda) \times B_n(\lambda) \qquad \text{for } M \in M_{m,n}^{h.w.} \qquad \lambda = (\lambda_i = m_{ii})$$

$$\therefore \quad M_{m,n} \cong \coprod_{\substack{\lambda:\text{par.} \\ \ell(\lambda) \leq m,n}} B_m(\lambda) \times B_n(\lambda)$$

This implies
$$S_q(m,n) \cong \bigoplus_{\substack{\lambda:\text{par.} \\ \ell(\lambda) \leq m,n}} V_m(\lambda) \otimes V_n(\lambda)$$

<u>Rmk</u> $\exists$ an explicit isomorphism of $(\mathfrak{gl}_m \oplus \mathfrak{gl}_n)$-crystals

$$M_{m,n} \longrightarrow \coprod_{\substack{\lambda:\text{par.} \\ \ell(\lambda) \leq m,n}} B_m(\lambda) \times B_n(\lambda)$$

$$M \longmapsto (P_M, Q_M)$$

RSK correspondence

insertion

recording

The case of $\Lambda(\mathbb{C}^m \otimes \mathbb{C}^n)$

$\Lambda_q(m,n)$ = the assoc. K-alg. gen. by X_{ij} $1 \leq i \leq m,\ 1 \leq j \leq n.$

subject to the following relations

$$X_{jk} X_{ik} = -q^{-1} X_{ik} X_{jk} \qquad \text{if } j > i$$

$$X_{il} X_{ik} = q X_{ik} X_{il} \qquad \text{if } l > k$$

$$X_{ij} X_{kl} = X_{kl} X_{ij} \qquad \text{if } i < k,\ j > l$$

$$X_{ij} X_{kl} - X_{kl} X_{ij} = (q^{-1} - q) X_{ik} X_{jl} \qquad \text{if } i < k,\ j < l.$$

$$X_{ij}^2 = 0$$

: a q-analogue of $\Lambda(\mathbb{C}^m \otimes \mathbb{C}^n)$

$\exists\ (U_q(\mathfrak{gl}_m), U_{\tilde{q}}(\mathfrak{gl}_n))$ - action on $\Lambda_q(m,n)$ $\quad (\tilde{q} = -q^{-1})$

$$U_q(\mathfrak{gl}_n) \longrightarrow U_{-q^{-1}}(\mathfrak{gl}_n)$$
$$q \longmapsto -q^{-1}$$
$$x_i \longmapsto x_i \quad (x = e, f)$$
$$k_i^{\pm 1} \longmapsto k_i^{\pm 1}$$

$M'_{m,n}(\mathbb{Z}_2) \ni M = (m_{ij}) \qquad X^M := \overrightarrow{\prod} X_{ij}^{m_{ij}}$

$\mathcal{L}' = \bigoplus A_0 X^M \qquad B' = \{ X^M \ (\mathrm{mod}\ q\mathcal{L}') \}$

$(\mathcal{L}', B')$: a crystal base of $\Lambda_q(m,n)$

Similarly, we have

$$M'_{m,n} \cong \coprod_{\substack{\lambda:\text{par.} \\ \ell(\lambda)\leq m \\ \ell(\lambda')\leq n}} B_m(\lambda)\times B_n(\lambda')$$

This implies $\Lambda_q(m,n) \cong \bigoplus_{\substack{\lambda:\text{par.} \\ \ell(\lambda)\leq m \\ \ell(\lambda')\leq n.}} V_m(\lambda)\otimes V_n(\lambda')$

<u>Rmk</u> As in $S_q(m,n)$, $\Lambda_q(m,n)$ can be viewed as a q-analogue of

$\underline{\mathcal{U}(\mathfrak{n}^-)}$ where $\mathfrak{gl}(m|n)^- = \mathfrak{n}^- \oplus \mathfrak{gl}(m|n)_{\bar{0}}$

$\cong \Lambda(\mathbb{C}^{m*}\otimes\mathbb{C}^n)$. $\mathfrak{gl}(m|n)$: a general linear Lie superalg.