Non-vanishing mod p of derived Hecke algebra of the multiplicative group over number field

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This work is joint with Dohyeong Kim.

Let us follow the paper of Venkatesh to define the derive Hecke algebra.

- F: a number field
- $\mathbf{G}$: a reductive algebraic group over F.
- K: an open compact subgroup of $\mathbf{G}(\mathbb{A}_F^{(\infty)})$ of form

$$\prod_{v\text{: prime of } F} K_v.$$

- R: a commutative ring with unity.
- v: a prime of F such that $N(v)$ is invertible in R.
- F_v: the completion of F at v.

Definition (Abstract Hecke algebra)

The **abstract Hecke algebra** at v with coefficients in R is defined by

$$\mathcal{H}_v^0 := \mathrm{Hom}(R[\mathbf{G}(F_v)/K_v], R[\mathbf{G}(F_v)/K_v]),$$

where Hom is taken in a suitable category of $R[\mathbf{G}(F_v)]$-modules.

Let S_∞ be the quotient of

$$\mathbf{G}(F \otimes_{\mathbb{Q}} \mathbb{R})$$

by a maximal compact connected subgroup of itself.

Notation

Let us define a space Y_K as follows:

$$Y_K := \mathbf{G}(F)\backslash S_\infty \times \mathbf{G}(\mathbb{A}_F^{(\infty)})/K.$$

Y_K is homeomorphic to a finite union of locally symmetric spaces.

Note

Singular cohomology groups

$$H^j(Y_K, R)$$

of the space Y_K are $\mathcal{H}_v^0$-modules.

Venkatesh considered a generalization of abstract Hecke algebra:

Definition (Abstract derived Hecke algebra)

The **abstract derived Hecke algebra** at v with coefficients in R is defined by

$$\mathcal{H}_v^\bullet := \mathrm{Ext}^\bullet(R[\mathbf{G}(F_v)/K_v], R[\mathbf{G}(F_v)/K_v]),$$

where $\mathrm{Ext}^\bullet$ is taken in a suitable category of $R[\mathbf{G}(F_v)]$-modules.

$\mathcal{H}_v^\bullet$ acts on the cohomology ring

$$H^\bullet(Y_K, R)$$

of the space Y_K in a graded fashion.

Let us write down the derived action explicitly by following Venkatesh.
Let us choose a double coset

$$z = K_v g_z K_v \in K_v \backslash \mathbf{G}(F_v)/K_v.$$

Let $K_z \subset K$ be the inverse image of

$$K_v \cap g_z K_v g_z^{-1} \subset K_v$$

under the projection $K \to K_v$ to the v-component.

Let us define a map as follows:

$$[z] : Y_{K_z} \to Y_K,\ (g_\infty, g^{(\infty)}) \mapsto (g_\infty, g^{(\infty)} g_z).$$

$[z]$ is a well-defined continuous map.

Let $K_{z,1,v}$ be a subgroup of $K_{z,v}$ such that the inflation map

$$\inf : H^\bullet(K_{z,v}/K_{z,1,v}, R) \to H^\bullet(K_{z,v}, R)$$

is an isomorphism. Let $K_{z,1} \subset K_z$ be the inverse image of $K_{z,v,1}$ under the projection map to the v-component:

$$K_z \to K_{z,v}$$

Let us consider the natural projection

$$\pi : Y_{K_{z,1}} \mapsto Y_{K_z}.$$

Note that this is a principal $\frac{K_{z,v}}{K_{z,1,v}}$-principal bundle, so there is a homotopy class

$$\left[Y_{K_z} \to \mathcal{B}\frac{K_{z,v}}{K_{z,1,v}} \right]$$

of maps corresponding to π. Taking $H^\bullet(-, R)$ and compositing $\inf^{-1}$, we obtain the following map of R-graded algebras:

$$\langle \cdot \rangle : H^\bullet(K_{z,v}, R) \xrightarrow[\cong]{\inf^{-1}} H^\bullet\left(K_{z,v}/K_{z,1,v}, R\right) \to H^\bullet(Y_{K_z}, R).$$

Let $z \in K_v \backslash \mathbf{G}(F_v)/K_v$ and $\alpha \in H^\bullet(K_{z,v}, R)$.

Let us define $h_{z,\alpha}$ by the following composition:

$$h_{z,\alpha} : H^\bullet(Y_K, R) \xrightarrow{[z]^*} H^\bullet(Y_{K_z}, R) \xrightarrow{-\cap\langle\alpha\rangle} H^\bullet(Y_{K_z}, R) \to H^\bullet(Y_K, R),$$

where the last map is the pushforward of the canonical projection

$$Y_{K_z} \to Y_K.$$

Let (z, α) be the image of the element

$$(0, \cdots, 0, \alpha, 0, \cdots, 0) \ (\textit{supported at } z)$$

under the following R-module isomorphism:

$$\bigoplus_{z \in K_v \backslash \mathbf{G}(F_v)/K_v} H^\bullet(K_{z,v}, R) \xrightarrow{\cong} \mathcal{H}_v^\bullet.$$

Then, the action of $\sum_i r_i(z_i, \alpha_i) \in \mathcal{H}_v^\bullet$ on $H^\bullet(Y_K, R)$ is given by

$$\sum_i r_i h_{z_i,\alpha_i},$$

Example (Hecke operator for modular forms)

Let us set

- $F = \mathbb{Q}$, $\mathbf{G} = \mathrm{GL}_{2/\mathbb{Q}}$.
- $K = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathbf{G}(\widehat{\mathbb{Z}}) : c, d-1 \in N\widehat{\mathbb{Z}} \right\}$, where $N > 0$ is an integer.
- q: a prime of $\mathbb{Q}$ s.t. $q \nmid N$.
- $z = K_q \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} K_q$.
- $\alpha \in H^0(K_{z,q}, \mathbb{C}) \cong \mathbb{C} \ni \alpha$.

In this case, Y_K is homotopically equivalent to the modular curve $Y_1(N)$. Let ω be the Eichler Shimura isomorphism:

$$\omega : S_2(\Gamma_1(N)) \oplus \overline{S_2(\Gamma_1(N))} \xrightarrow{\cong} H^1_{\mathrm{par}}(Y_1(N), \mathbb{C}) \subset H^1(Y_1(N), \mathbb{C})$$

Then, for a cusp form $f \in S_2(\Gamma_1(N))$, we have

$$h_{z,\alpha}\omega(f) = \alpha\omega(T_q f).$$

Venkatesh proves that the action of $\mathcal{H}_w^\bullet$ and $\mathcal{H}_v^\bullet$ commute for any prime $w \neq v$ of F.

Definition (Derived Hecke algebra)

Let us define the **derived Hecke algebra** $\mathbb{T}^\bullet$ by the graded R-subalgebra of

$$\mathrm{End}_R(H^\bullet(Y_K, R))$$

generated by the action of $\mathcal{H}_v^\bullet$ on

$$H^\bullet(Y_K, R)$$

for primes v of F s.t. $N(v)$ is invertible in R.

For $j \geq 0$, let $\mathbb{T}^j$ be the R-submodule of degree j shifting actions of $\mathbb{T}^\bullet$.

Using $\mathbb{T}^\bullet$, Venkatesh explain the phenomenon called **spectral degeneracy**:

Spectral degeneracy

Spectral degeneracy refers the phenomenon that the same Hecke eigensystem can occur in several different cohomological degrees.

If Y_K is not an algebraic variety, spectral degeneracy often happens. Assume the followings:

(A) $\mathbb{T}^j$ is non-vanishing for some $j > 0$.

(B) The action of $\mathbb{T}^0$ commute with the one of $\mathbb{T}^j$.

Then one can observe that the spectral degeneracy occurs.

Specialization

From now on, let us set

- $\mathbf{G} = \mathbb{G}_{\mathrm{m}/F}$.
- $\mathfrak{N}$: an integral ideal of F and

$$K = K(\mathfrak{N}) := \left(\prod_{v|\mathfrak{N}} 1 + \mathfrak{N}O_v\right) \times \prod_{v\nmid\mathfrak{N}} O_v^\times.$$

- $S_\infty = (F \otimes_{\mathbb{Q}} \mathbb{R})^\times / U$, where

$$U := \{x \in (F \otimes_{\mathbb{Q}} \mathbb{R})_+^\times : |x^\tau| = 1 \text{ for any infinite places } \tau \text{ of } F\}.$$

- p: a rational prime.
- $R = \mathbb{F}_p$.

Here is our main result:

Theorem (Kim--Kwon, 2025^+)

- (A) *holds,*
- (B) *holds for all but finitely many* p*,*

where

(A) $\mathbb{T}^j$ *is non-vanishing for some* $j > 0$.

(B) *The action of* $\mathbb{T}^0$ *commute with the one of* $\mathbb{T}^j$.

From now on, we assume that $v \nmid \mathfrak{N}$ and $p \nmid N(v)$ because

Note

If $v \mid \mathfrak{N}$, then the action of $\mathcal{H}_v^j$ at v is trivial for $j > 0$.

Let us set

$$Y_{K(\mathfrak{N})} = Y(\mathfrak{N}).$$

Let us consider the natural projection

$$\pi_{v,\mathfrak{N}} : Y(v\mathfrak{N}) \to Y(\mathfrak{N}).$$

Let κ_v be the residue field of v. Note that this is a principal $\kappa_v^\times$-bundle, so there is a homotopy class

$$[Y(\mathfrak{N}) \to \mathcal{B}\kappa_v^\times]$$

of maps corresponding to $\pi_{v,\mathfrak{N}}$. Taking $H^\bullet(-, \mathbb{F}_p)$ and compositing $\inf^{-1}$, we obtain the following map of $\mathbb{F}_p$-graded algebras:

$$\langle \cdot \rangle : H^\bullet(O_v^\times, \mathbb{F}_p) \xrightarrow[\cong]{\inf^{-1}} H^\bullet(\kappa_v^\times, \mathbb{F}_p) \to H^\bullet(Y(\mathfrak{N}), \mathbb{F}_p).$$

For a coset,

$$z = g_z O_v^\times \in F_v^\times / O_v^\times,$$

let us define a map as follows:

$$[z] : Y(\mathfrak{N}) \to Y(\mathfrak{N}),\ (g_\infty, g^{(\infty)}) \mapsto (g_\infty, g^{(\infty)} g_z).$$

This is a well-defined continuous map. We have the following $\mathbb{F}_p$-algebra isomorphism:

$$R[F_v^\times / O_v^\times] \otimes_{\mathbb{F}_p} H^\bullet(O_v^\times, \mathbb{F}_p) \xrightarrow{\cong} \mathcal{H}_v^\bullet.$$

Let us denote by (z, α) the image of $z \otimes \alpha$ under the isomorphism above.

Let us define $h_{z,\alpha}$ the following composition:

$$h_{z,\alpha} : H^\bullet(Y_K, \mathbb{F}_p) \xrightarrow{[z]^*} H^\bullet(Y_K, \mathbb{F}_p) \xrightarrow{-\cup\langle\alpha\rangle} H^\bullet(Y_K, \mathbb{F}_p).$$

Then, the action of $\sum_i r_i(z_i, \alpha_i) \in \mathcal{H}_v^\bullet$ on $H^\bullet(Y_K, \mathbb{F}_p)$ is given by

$$\sum_i r_i h_{z_i, \alpha_i}.$$

Put

$$T_F := (F \otimes_{\mathbb{Q}} \mathbb{R})_+ / U$$

and

$$E(\mathfrak{M}) := F_+ \cap K(\mathfrak{M}) \subset O_F^\times .$$

Then, we have the following homeomorphism:

$$\coprod_{a \in \mathrm{Cl}_F^+(\mathfrak{M})} E(\mathfrak{M}) \backslash T_F \xrightarrow{\bigoplus_a \iota_a} Y(\mathfrak{M}),$$

where $\mathrm{Cl}_F^+(\mathfrak{M})$ is the narrow ray class group of modulus $\mathfrak{M}$.
We have the following maps of principal bundles:

$$\begin{array}{ccccc} T_F & \longrightarrow & E(v\mathfrak{N}) \backslash T_F & \xrightarrow{\iota_a} & Y(v\mathfrak{N}) \\ \downarrow & & \downarrow & & \downarrow \pi_{v,\mathfrak{N}} \\ E(\mathfrak{N}) \backslash T_F & \xrightarrow{=} & E(\mathfrak{N}) \backslash T_F & \xrightarrow{\iota_a} & Y(\mathfrak{N}). \end{array}$$

Let us define a map as follows:

$$i_{v,\mathfrak{N}} : E(\mathfrak{N})/E(v\mathfrak{N}) \to \kappa_v^\times,\ \varepsilon \mapsto \varepsilon \mod v.$$

Note that the maps between principal bundles

$$\begin{array}{ccccc} T_F & \longrightarrow & E(v\mathfrak{N})\backslash T_F & \xrightarrow{\iota_a} & Y(v\mathfrak{N}) \\ \downarrow & & \downarrow & & \downarrow \pi_{v,\mathfrak{N}} \\ E(\mathfrak{N})\backslash T_F & \xrightarrow{=} & E(\mathfrak{N})\backslash T_F & \xrightarrow{\iota_a} & Y(\mathfrak{N}). \end{array}$$

is equivariant with respect to the maps

$$E(\mathfrak{N}) \xrightarrow{\mod E(v\mathfrak{N})} \frac{E(\mathfrak{N})}{E(v\mathfrak{N})} \xrightarrow{i_{v,\mathfrak{N}}} \kappa_v^\times.$$

Therefore, we obtain the following homotopy commutative diagram:

$$\begin{array}{ccccc} E(\mathfrak{N})\backslash T_F & \xrightarrow{=} & E(\mathfrak{N})\backslash T_F & \xrightarrow{\iota_a} & Y(\mathfrak{N}) \\ \downarrow \cong & & \downarrow & & \downarrow \\ \mathcal{B}E(\mathfrak{N}) & \xrightarrow{\mathcal{B}\mod E(v\mathfrak{N})} & \mathcal{B}\frac{E(\mathfrak{N})}{E(v\mathfrak{N})} & \xrightarrow{\mathcal{B}i_{v,\mathfrak{N}}} & \mathcal{B}\kappa_v^\times. \end{array}$$

Taking $H^\bullet(-,\mathbb{F}_p)$ to the following diagram

$$\begin{array}{ccccc}
E(\mathfrak{N})\backslash T_F & \xrightarrow{=} & E(\mathfrak{N})\backslash T_F & \xrightarrow{\iota_a} & Y(\mathfrak{N}) \\
\downarrow{\cong} & & \downarrow & & \downarrow \\
\mathcal{B}E(\mathfrak{N}) & \xrightarrow{\mathcal{B}\mathrm{mod}\ E(v\mathfrak{N})} & \mathcal{B}\frac{E(\mathfrak{N})}{E(v\mathfrak{N})} & \xrightarrow{i_{v,\mathfrak{N}}} & \mathcal{B}\kappa_v^\times ,
\end{array}$$

we obtain the following commutative diagram:

$$\begin{array}{ccccc}
H^\bullet(E(\mathfrak{N})\backslash T_F,\mathbb{F}_p) & \xleftarrow[=]{} & H^\bullet(E(\mathfrak{N})\backslash T_F,\mathbb{F}_p) & \xleftarrow[\iota_a^*]{} & H^\bullet(Y(\mathfrak{N}),\mathbb{F}_p) \\
{\cong}\uparrow & & \uparrow & & \uparrow \\
H^\bullet(E(\mathfrak{N}),\mathbb{F}_p) & \xleftarrow[(\mathrm{mod}\ E(v\mathfrak{N}))^*]{} & H^\bullet(\frac{E(\mathfrak{N})}{E(v\mathfrak{N})},\mathbb{F}_p) & \xleftarrow[i_{v,\mathfrak{N}}^*]{} & H^\bullet(\kappa_v^\times,\mathbb{F}_p) \\
 & & & & \mathrm{inf}^{-1}\uparrow \\
 & & & & H^\bullet(O_v^\times,\mathbb{F}_p)
\end{array}$$

In conclude, for $z = g_z O_v^\times \in F_v^\times / O_v^\times$ and $\alpha \in H^\bullet(O_v^\times, \mathbb{F}_p)$,

$$\iota_a^*(h_{z,\alpha} c) = \iota_{g_z a}^*(c) \cup i_{v,\mathfrak{N}}^*(\alpha)$$

for $c \in H^\bullet(Y(\mathfrak{N}), \mathbb{F}_p)$ and $a \in \mathrm{Cl}_F^+(\mathfrak{N})$, i.e., the following diagram

$$\begin{array}{ccc} H^\bullet(Y(\mathfrak{N}), \mathbb{F}_p) & \xrightarrow{h_{z,\alpha}} & H^\bullet(Y(\mathfrak{N}), \mathbb{F}_p) \\ \downarrow \iota_{g_z a}^* & & \downarrow \iota_a^* \\ H^\bullet(E(\mathfrak{N}), \mathbb{F}_p) & \xrightarrow{-\cup i_{v,\mathfrak{N}}^*(\alpha)} & H^\bullet(E(\mathfrak{N}), \mathbb{F}_p), \end{array}$$

commutes, where

$$i_{v,\mathfrak{N}} : E(\mathfrak{N}) \to O_v^\times$$

is the inclusion.

From this formula, we obtain the following result:

Theorem (Kim--Kwon, 2025^+)

(A) *The map*

$$\Psi : \mathbb{T}^1 \otimes_{\mathbb{T}^0} H^0(Y(\mathfrak{N}), \mathbb{F}_p) \to H^1(Y(\mathfrak{N}), \mathbb{F}_p), \; h \otimes c \mapsto hc$$

is an isomorphism if and only if

$$p \nmid t_F \cdot \# \left(O_F^\times / E(\mathfrak{N}) \right) \text{ or } p \mid t_F \;\&\; p \mid \# \left(O_F^\times / E(\mathfrak{N}) \right),$$

where t_F is the order of the torsion subgroup of $O_F^\times$.

(B) $\mathbb{T}^\bullet$ *is a graded $\mathbb{T}^0$-algebra.*

Corollary (Kim--Kwon, 2025^+)

If Ψ is non-vanishing, then the same Hecke eigensystem occur in

$$H^0(Y(\mathfrak{N}), \overline{\mathbb{F}}_p) \text{ and } H^1(Y(\mathfrak{N}), \overline{\mathbb{F}}_p).$$

Remark

- $t_p \neq r$ if and only if

$$p \mid t_F \text{ and } p \nmid \#\left(O_F^\times / E(\mathfrak{N})\right)$$

or

$$p \nmid t_F \text{ and } p \mid \#\left(O_F^\times / E(\mathfrak{N})\right).$$

- The number of p such that $t_p \neq r$ is finite.

Let us prove Theorem A.

Observation

By straightforward computation, we observe that

$$\dim_{\mathbb{F}_p}\mathrm{Codom}(\Psi) = |\mathrm{Cl}_F^+(\mathfrak{N})| \cdot \mathrm{rk}_{\mathbb{Z}} O_F^\times,$$
$$\dim_{\mathbb{F}_p}\mathrm{Dom}(\Psi) = \dim_{\mathbb{F}_p}\mathrm{Im}(\Psi) = |\mathrm{Cl}_F^+(\mathfrak{N})| \cdot t_p,$$

where

$$t_p := \dim_{\mathbb{F}_p}\left(\sum_{\substack{v:\ \text{prime of } F \\ p\nmid N(v),\ v\nmid\mathfrak{N}}} i_{v,\mathfrak{N}}^* \mathrm{Hom}(\kappa_v^\times, \mathbb{F}_p)\right) \leq \dim_{\mathbb{F}_p}\mathrm{Hom}(E(\mathfrak{N}), \mathbb{F}_p)$$

and $i_{v,\mathfrak{N}}$ is the canonical projection

$$E(\mathfrak{N}) \to \kappa_v^\times,$$

so that Ψ is injective.

Grunwald--Wang theorem

Let $\xi \in F^\times$. If $\xi \in (F_v^\times)^p$ for all but finitely many primes v of F, then

$$\xi \in (F^\times)^p.$$

From this theorem, we obtain the following:

Lemma

$$\begin{aligned} t_p &= \dim_{\mathbb{F}_p} \mathrm{Hom}(E(\mathfrak{N}), \mathbb{F}_p) \\ &= \dim_{\mathbb{F}_p} \mathrm{Hom}(O_F^\times, \mathbb{F}_p) - \dim_{\mathbb{F}_p} \mathrm{Hom}(O_F^\times / E(\mathfrak{N}), \mathbb{F}_p). \end{aligned}$$

Proof of $t_p = \mathrm{rk}_{\mathbb{Z}} O_F^\times$

From the Lemma, we know that $t_p = r$ if and only if

$$p \nmid t_F \cdot \# \left(O_F^\times / E(\mathfrak{N}) \right)$$

or p divides both t_F and $\# \left(O_F^\times / E(\mathfrak{N}) \right)$.

Remark

$H^\bullet(O_v^\times, \mathbb{F}_p)$ is isomorphic to

$$\mathbb{F}_p[\alpha, \beta]/(\alpha^2),$$

where α and β map to a generator of $H^1(O_v^\times, \mathbb{F}_p)$ and $H^2(O_v^\times, \mathbb{F}_p)$, resp. Based on this fact, we obtain that

$$\mathbb{T}^\bullet \text{ is generated by } \mathbb{T}^1, \mathbb{T}^2 \text{ as a } \mathbb{T}^0\text{-module.}$$

Question

Is the map

$$\Phi : \mathbb{T}^2 \otimes_{\mathbb{T}^0} H^0(Y(\mathfrak{N}), \mathbb{F}_p) \to H^2(Y(\mathfrak{N}), \mathbb{F}_p), \ h \otimes c \mapsto hc$$

is non-vanishing?

Thank you!