Algebra camp 2025

Cluster algebras and Poisson geometry

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Plan

- Motivation for cluster algebras from total positivity;
- An example of a cluster structure in $\mathrm{SL}_3(\mathbb{C})$;
- Laurent phenomenon;
- Connection between cluster algebras and Poisson geometry;
- How to construct a cluster structure using Poisson geometry;
- A class of Belavin-Drinfeld Poisson brackets;
- Program on constructing cluster structures compatible with Belavin-Drinfeld brackets;
- A list of some open problems in cluster theory;
- (time permissible) $\mathcal{A}$-$\mathcal{X}$ cluster duality and quantization

A totally positive story

Def. An $n \times n$ matrix A is *totally positive* if all its minors are positive.
(consider matrices with real entries)

Def. A *test* for total positivity is a minimal collection of minors such that if they are positive on a matrix A, then A is totally positive.

Example. $n = 2$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad ad = bc + \det A$$

Two tests:

$$\{a, b, c, \det A\} \text{ —— } \{d, b, c, \det A\}$$

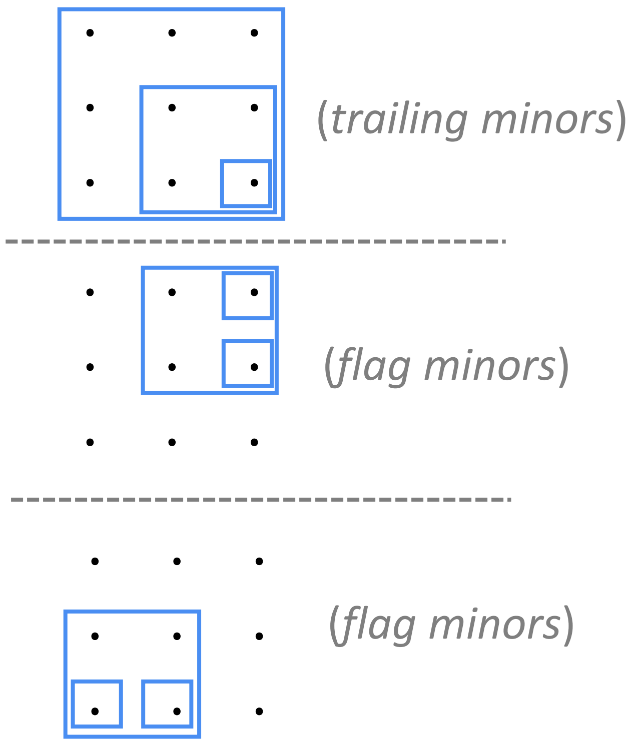
(organize the tests into an *exchange graph*; the edge represents the *exchange relation*: one can replace a with d and obtain a new test)

Example. $n = 3$ $\quad X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$

An example of a test:

$$\left\{\det X,\ \det\begin{bmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{bmatrix},\ x_{33},\ x_{13},\ \det\begin{bmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{bmatrix},\ x_{23},\ x_{31},\ \det\begin{bmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix},\ x_{32}\right\}$$

(schematic representation)



Can never be replaced with another minor to yield a new test (future *frozen variables*)

Can be replaced with another minor to yield a new test (future *cluster variables*)

In this test, the minor

$$\det\begin{bmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{bmatrix}$$

cannot be replaced with another minor of X to yield a test; however, it appears in some other tests where it *can* be replaced with a minor.

Example continued. Some exchange relations. A new test:

$$\left\{\det X,\ \det\begin{bmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{bmatrix}, x_{33},\ x_{13},\ \det\begin{bmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{bmatrix}, x_{23},\ x_{31},\ \det\begin{bmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix}, x_{32}\right\}$$

$$\left\{\det X,\ \det\begin{bmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{bmatrix}, x_{33},\ x_{13},\ \det\begin{bmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{bmatrix}, x_{23},\ x_{31},\ \det\begin{bmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix}, \det\begin{bmatrix} x_{21} & x_{23} \\ x_{31} & x_{33} \end{bmatrix}\right\}$$

(replaces x_{32})

$$x_{32} \cdot \det\begin{bmatrix} x_{21} & x_{23} \\ x_{31} & x_{33} \end{bmatrix} = \det\begin{bmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{bmatrix} x_{31} + \det\begin{bmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix} x_{33}$$ (short Plücker relation)

(replaces x_{33})

$$x_{33} \cdot x_{22} = \det\begin{bmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{bmatrix} + x_{23}x_{32}$$

(an exchange relation for x_{23} can be obtained via transposing X in the exchange relation for x_{32})

Why do we get a new test? If all minors in the test are already known to be positive and we have not yet checked the minor x_{32}, we see that it is positive if and only if $\det\begin{bmatrix} x_{21} & x_{23} \\ x_{31} & x_{33} \end{bmatrix}$ is positive (see the exchange relation); hence, a new collection of minors is a test if and only if the current one is a test.

An aesthetical issue

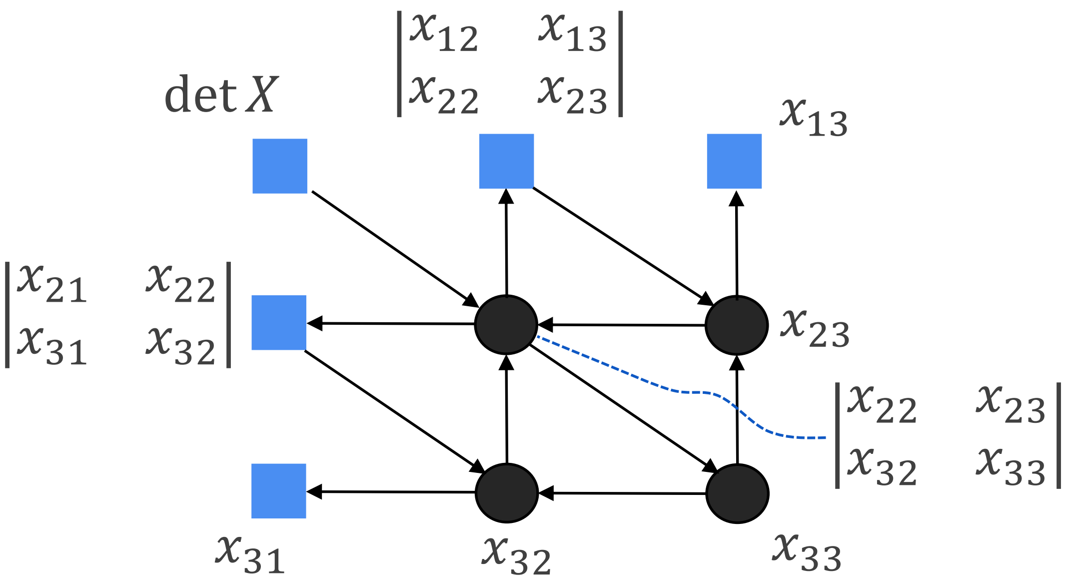
The exchange graph is not regular.

(in other words, different tests have different numbers of minors that can be replaced)

(Seed/cluster vs Extended seed/cluster: a cluster contains only cluster variables, whereas an extended cluster contains both cluster and frozen variables. Not much of a difference, b/c frozen variables never change, and they are always ‘in the background’)

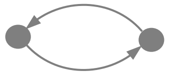
Question. Is there a framework that makes the exchange graph regular?

Answer (by Fomin & Zelevinsky). Cluster algebras.



(det is dropped for saving space)

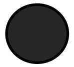
(the quiver encodes the exchange relations)

Def. A *quiver* is a directed multigraph with no loops and 2-cycles. ( is not allowed)

Def. Collection of functions + Quiver.

- Collection of functions: *extended cluster*
- Collection of functions + Quiver: *extended seed*

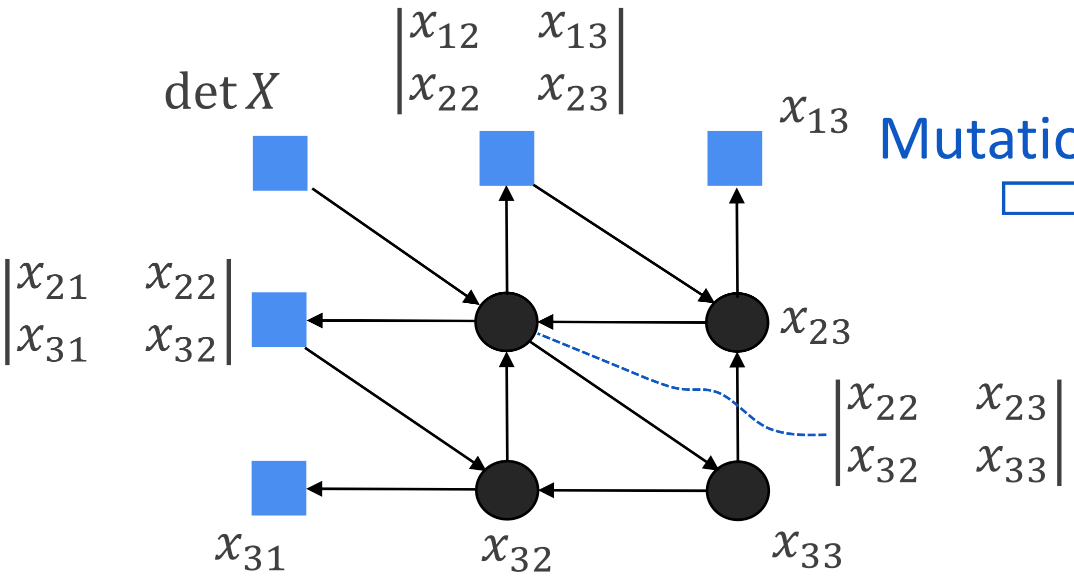
Def.  *frozen vertex* (the attached function is a *frozen variable*)

 *mutable vertex* (the attached function is a *cluster variable*)

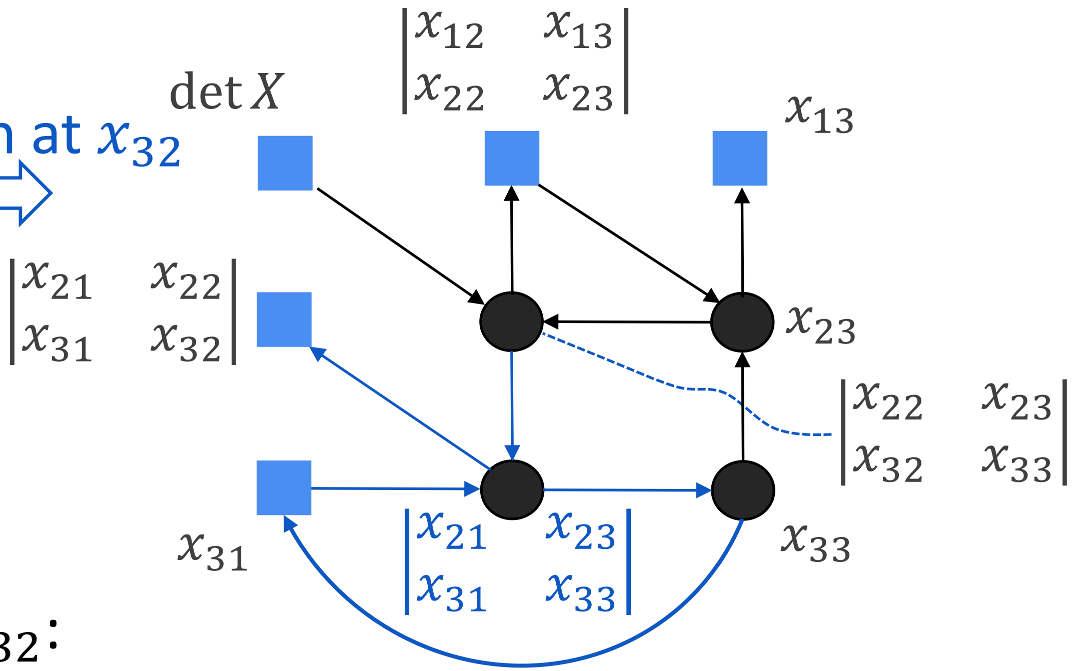
Mutation (by example)

A *mutation* is an involutive operation on (extended) seeds:

- Replaces a chosen cluster variable with a new one;
- Updates the quiver.



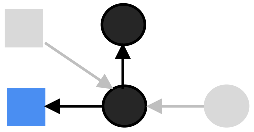
Mutation at x_{32}

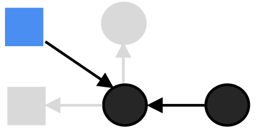


A *mutation* (or *exchange*) *relation* for x_{32}:

$$x_{32} \cdot \det \begin{bmatrix} x_{21} & x_{23} \\ x_{31} & x_{33} \end{bmatrix} = \det \begin{bmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{bmatrix} x_{31} + \det \begin{bmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix} x_{33}$$

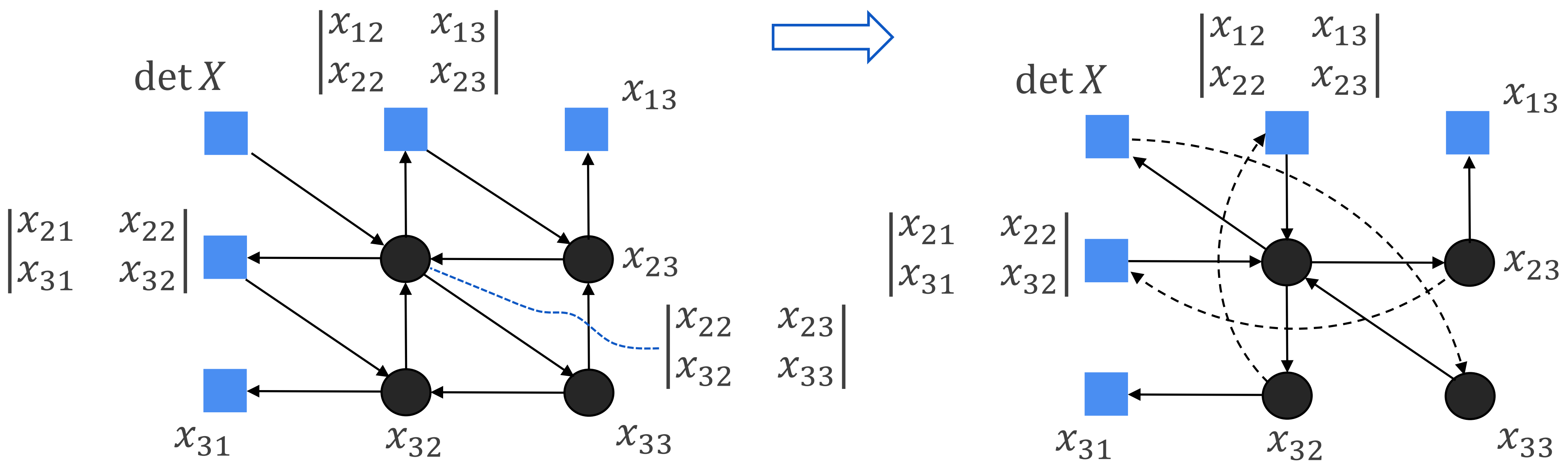
A new cluster variable that replaces x_{32}





New tests for total positivity

Mutation at $\det\begin{bmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{bmatrix}$



(some arrows are dashed only for readability)

Cluster theory produces a new test; however, the new function is not a minor!

$$\det\begin{bmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{bmatrix} \cdot \left(x_{32} \det\begin{bmatrix} x_{11} & x_{13} \\ x_{21} & x_{23} \end{bmatrix} - x_{31} \det\begin{bmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{bmatrix} \right) = $$

$$= x_{23}x_{32} \det X + \det\begin{bmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{bmatrix} \det\begin{bmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix} x_{33}$$

Not a short Plücker relation! But it is a mutation relation.

(in total, there are 16 cluster variables, 2 of which are not minors of X; there are 50 distinct seeds)

Algebras and Laurent phenomenon

Let C be a cluster structure.

Def. *Cluster algebra* $\mathcal{A}(C) := \mathbb{Z}[$ all cluster and frozen variables from C].

Def. *Upper cluster algebra*

$$\bar{\mathcal{A}}(C) := \bigcap_{\text{all } \mathbf{x} \text{ in C}} \mathbb{Z}\left[x_1^{\pm 1}, \dots, x_N^{\pm 1}, x_{N+1}, \dots, x_{N+M} \,\middle|\, x_i \in \boldsymbol{x}\right].$$

Theorem (Laurent Phenomenon). $\mathcal{A}(C) \subseteq \bar{\mathcal{A}}(C)$.

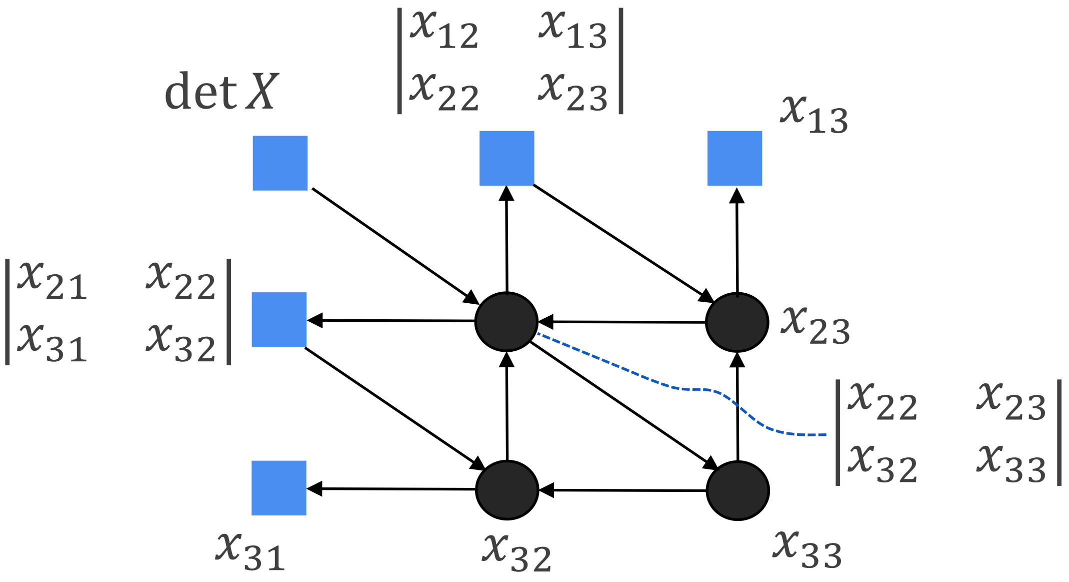
(Fomin, Zelevinsky, 2001).

That is, any cluster variable can be written as a Laurent polynomial in terms of any cluster with coefficients in frozen variables.

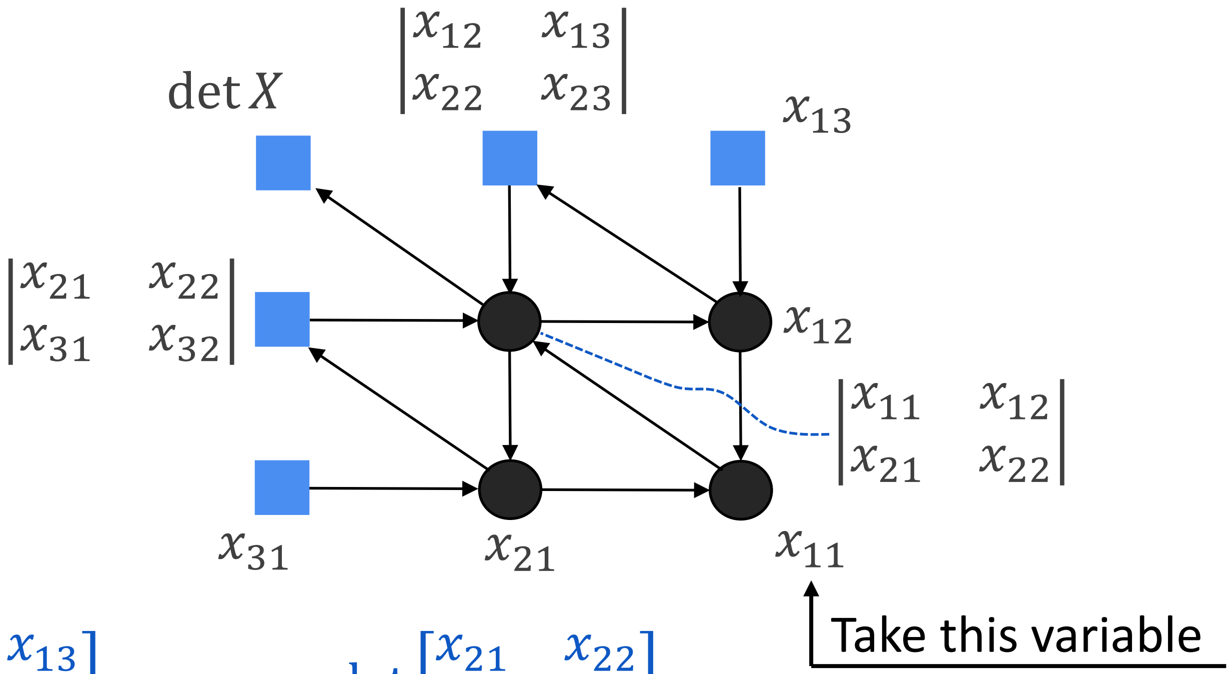
Moreover, the Laurent polynomials in the theorem can be chosen with nonnegative coefficients.

(M. Gross, P. Hacking, S. Keel, M. Kontsevich, 2015)

An illustration of the Laurent phenomenon



Some mutation equivalent seed



$$x_{11} = \frac{\det X}{\det\begin{bmatrix}x_{22} & x_{23}\\ x_{32} & x_{33}\end{bmatrix}} + \frac{\det\begin{bmatrix}x_{21} & x_{22}\\ x_{31} & x_{32}\end{bmatrix}\det\begin{bmatrix}x_{12} & x_{13}\\ x_{22} & x_{23}\end{bmatrix} x_{33}}{x_{23}x_{32}\det\begin{bmatrix}x_{22} & x_{23}\\ x_{32} & x_{33}\end{bmatrix}} + \frac{x_{13}\det\begin{bmatrix}x_{21} & x_{22}\\ x_{31} & x_{32}\end{bmatrix}}{x_{23}x_{32}} + \frac{x_{31}\det\begin{bmatrix}x_{12} & x_{13}\\ x_{22} & x_{23}\end{bmatrix}}{x_{23}x_{32}} + \frac{x_{13}x_{31}\det\begin{bmatrix}x_{22} & x_{23}\\ x_{32} & x_{33}\end{bmatrix}}{x_{23}x_{32}x_{33}}$$

1) Frozen variables do not appear in the denominators;
2) No negative signs.

Program on cluster algebras

(Fomin, Zelevinsky, 2001): For every interesting variety V over a field $\mathbb{K}$ in Lie theory, there exists C such that $\mathbb{K}[V] = \mathcal{A}_{\mathbb{K}}(C)$.

C: defined in the field of rational functions of V

$\mathbb{K}[V]$: coordinate ring of V

$\mathcal{A}_{\mathbb{K}}(C) \coloneqq \mathcal{A}(C) \otimes_{\mathbb{Z}} \mathbb{K}$

$\bar{\mathcal{A}}_{\mathbb{K}}(C) \coloneqq \bar{\mathcal{A}}(C) \otimes_{\mathbb{Z}} \mathbb{K}$

Example. $\mathbb{K}[\mathrm{Mat}_{n\times n}] = \mathcal{A}_{\mathbb{K}}(C)$. ($n = 3$ is constructed on the previous slides; in fact, any coordinate function x_{ij} is a cluster variable in some cluster for this C)

$\mathrm{Mat}_{n\times n}$: matrices of size $n \times n$ over any field

However, most often we are only able to show $\mathcal{O}(V) = \bar{\mathcal{A}}_{\mathbb{K}}(C)$.

Question. How to construct a cluster structure in the first place?

Cluster algebras and Poisson geometry

Setup. $\mathcal{A} \coloneqq$ a commutative associative algebra over a field $\mathbb{K}$.

Def. A *Poisson bracket* $\{\ ,\ \}: \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is a skew-symmetric bilinear form that satisfies the Jacobi identity and the Leibniz rule in each slot.

Jacobi identity: $\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0$

Leibniz rule: $\{a \cdot b, c\} = a\{b, c\} + \{a, c\}b$

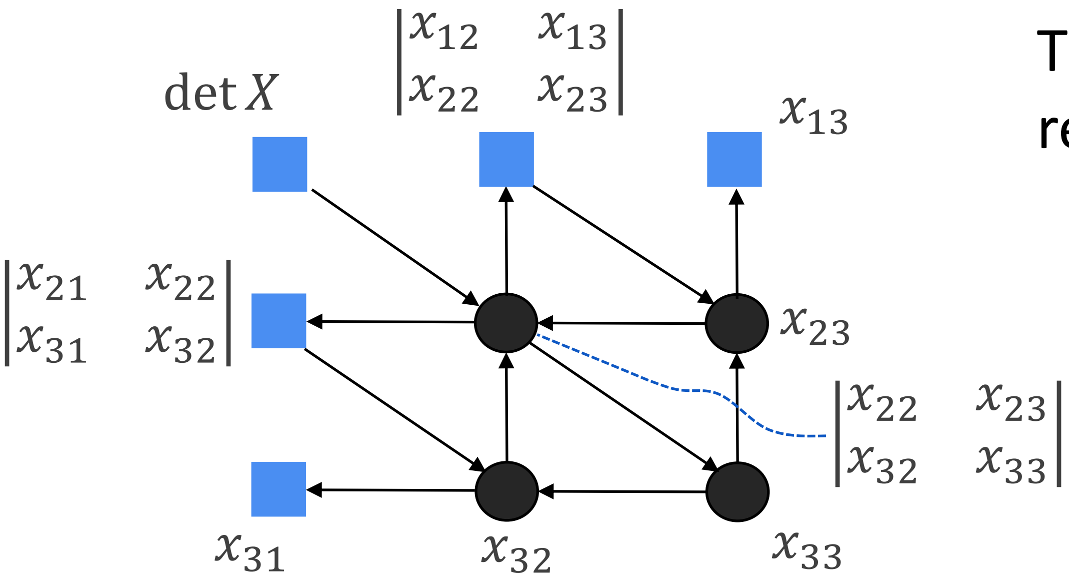
$\forall a, b, c \in \mathcal{A}$

Def. Elements $a, b \in \mathcal{A}$ are *log-canonical* if $\{a, b\} = \omega \cdot ab$, $\omega \in \mathbb{K}$.

A subset $S \subseteq \mathcal{A}$ is *log-canonical* if all elements in S are pairwise log-canonical.

Def. A cluster structure is *compatible* with a Poisson bracket if every extended cluster is log-canonical.

A remarkable observation (M. Gekhtman, M. Shapiro, A. Vainshtein, 2003)



This initial extended cluster is log-canonical with respect to the *standard* Poisson bracket $\{\,,\}_{\mathrm{st}}$ on GL_3.

(we explain later what standard means)

For example,

$$\{x_{31}, x_{32}\}_{\mathrm{st}} = x_{31}x_{32}, \quad \{x_{23}, x_{33}\}_{\mathrm{st}} = \frac{1}{2}x_{23}x_{33},$$

$$\left\{\det\begin{bmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{bmatrix}, x_{33}\right\}_{\mathrm{st}} = 0.$$

Even more is true:

- *Every* extended cluster is log-canonical;
- The frozen variables generate *Poisson prime ideals* in $\mathbb{C}[\mathrm{GL}_3]$:

$$\{f, g\} \in (f) \text{ for any frozen } f \text{ and any } g \in \mathbb{C}[\mathrm{GL}_3];$$

Geometrically, this is equivalent to saying that $\{X \in \mathrm{GL}_3 \mid f(X) = 0\}$ is a union of symplectic leaves of $\{\,,\}_{\mathrm{st}}$.

(B. Nguyen, K. Trampel, M. Yakimov, 2017)

How to verify the compatibility with a Poisson bracket?

Setup. $(\boldsymbol{x}, Q)$ the initial extended seed of some cluster structure,

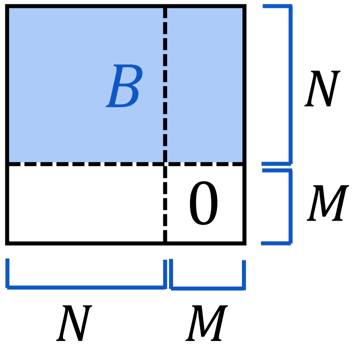
↑ quiver

$$\boldsymbol{x} := (x_1, \dots, x_N, x_{N+1}, \dots, x_{N+M}).$$

$x_1, \dots, x_N$: cluster variables; $x_{N+1}, \dots, x_{N+M}$: frozen variables

Exchange matrix B:

Adjacency matrix of Q =



Can remember $(\boldsymbol{x}, B)$ instead of $(\boldsymbol{x}, Q)$.

$\mathcal{F} := \mathbb{K}(x_1, \dots, x_{N+M})$;

$\{\,,\}: \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ a Poisson bracket.

Theorem. Assume that the mutable part of Q is connected. TFAE:

i) B has full rank and the cluster structure is compatible with $\{\,,\}$;

ii) $\boldsymbol{x}$ is log-canonical and $B\Omega = [\lambda I \quad 0]$ (compatibility equation)

where $\lambda \in \mathbb{K}^*$, $\Omega := (\omega_{ij})_{i,j=1}^{N+M}$, $\{x_i, x_j\} = \omega_{ij} x_i x_j$, $\omega_{ij} \in \mathbb{K}$.

(M. Gekhtman, M. Shapiro, A. Vainshtein, 2003)

How to construct a cluster structure?

Setup. A variety V over a field $\mathbb{K}$, $n := \dim V$.

field of rational functions on V

Empirical algorithm:

1) Introduce a Poisson bracket $\{\,,\}: \mathbb{K}(V) \times \mathbb{K}(V) \to \mathbb{K}(V)$;
2) Find a log-canonical family $\boldsymbol{x} := (x_1, \dots, x_n)$, $x_i \in \mathcal{O}(V)$;

   Set $\Omega := (\omega_{ij})_{i,j=1}^{n}$, $\{x_i, x_j\} = \omega_{ij} x_i x_j$, $\omega_{ij} \in \mathbb{K}$.
3) For frozen variables, choose those x_i that generate Poisson prime ideals.

   This also gives $n = N + M$ where $M =$ # frozen variables.
4) Solve the compatibility equation with respect to an $N \times (N + M)$ matrix B:

$$B\Omega = [I \;\; 0] \qquad (*)$$

   (if B has entries in $\mathbb{Q}$, can multiply both sides by a common denominator)

   Then $(\boldsymbol{x}, B)$ is the initial extended seed for some cluster structure C on V.

Remark. If $(*)$ has many solutions, can introduce extra structures (e.g., a toric action). This way one extracts a unique solution.

Belavin-Drinfeld classification

Setup. G is a simple complex Lie group, $\mathfrak{g} := \mathrm{Lie}(G)$, Π a set of simple roots,
$\mathfrak{h}$ the Cartan subalgebra, $< , >$ symmetric invariant nondegenerate form on $\mathfrak{g}$.

Def. A *Belavin-Drinfeld triple* (a *BD triple*) is $\mathbf{\Gamma} := (\Gamma_1, \Gamma_2, \gamma)$ where $\Gamma_1, \Gamma_2 \subset \Pi$ and $\gamma: \Gamma_1 \to \Gamma_2$ is a nilpotent isometry.

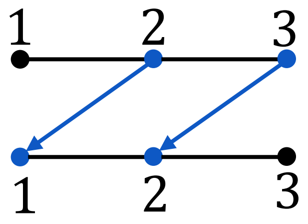
(that is, $\forall \alpha \in \Gamma_1 \exists m > 0 : \gamma^m(\alpha) \notin \Gamma_1$)

A *Belavin-Drinfeld quadruple* (a *BD quadruple*) is $\overline{\mathbf{\Gamma}} := (\mathbf{\Gamma}, R_0)$ where $R_0\colon \mathfrak{h} \to \mathfrak{h}$ is a linear map that satisfies

$$R_0 + R_0^* = \mathrm{id}$$
$$R_0(1-\gamma)(\alpha) = \alpha, \;\; \alpha \in \Gamma_1$$

Compatible cluster structures don't depend on R_0, but the Poisson brackets do.

Example. $\mathfrak{g} = \mathrm{sl}_4(\mathbb{C}),\; \Gamma_1 := \{2,3\},\; \Gamma_2 := \{1,2\},\; \gamma(2) := 1,\; \gamma(3) := 2.$



γ is an *isometry*: $\langle 2,3 \rangle = \langle 1,2 \rangle = \langle \gamma(1), \gamma(2) \rangle$

γ is *nilpotent*: $3 \overset{\gamma}{\mapsto} 2 \overset{\gamma}{\mapsto} 1 \notin \Gamma_1$

Belavin-Drinfeld classification

Def. A *Classical Yang-Baxter equation (CYBE)* is the equation for a linear map $R\colon \mathfrak{g} \to \mathfrak{g}$ of the form

$$[R(x), R(y)] = R([R(x), y] - [x, R^*(y)]), \;\; x, y \in \mathfrak{g}.$$

Theorem (Belavin-Drinfeld, 1982). Belavin-Drinfeld quadruples $\overline{\boldsymbol{\Gamma}} := (\Gamma_1, \Gamma_2, \gamma, R_0)$ parameterize the space of solutions of the CYBE.

$\overline{\boldsymbol{\Gamma}} := (\Gamma_1, \Gamma_2, \gamma, R_0)$ → Solution of CYBE R → A Poisson bracket $\{\,,\}_{\overline{\boldsymbol{\Gamma}}}$ on G

Formula for R: $(x \in \mathfrak{g})$

$$R(x) = \frac{1}{1-\gamma}\pi_>(x) - \frac{\gamma^*}{1-\gamma^*}\pi_<(x) + R_0\pi_0(x),$$

$\pi_>$: Projection onto $\mathfrak{b}_+$ (upper Borel); $\pi_<$: onto $\mathfrak{b}_-$; π_0: onto $\mathfrak{h}$

γ can be extended to a linear map $\gamma\colon \mathfrak{g} \to \mathfrak{g}$

For $\overline{\boldsymbol{\Gamma}} := (\emptyset, \emptyset, \emptyset \to \emptyset, R_0)$:

$$R_{\mathrm{st}}(x) = \pi_>(x) + R_0\pi_0(x).$$

Belavin-Drinfeld classification

Def. A *Classical Yang-Baxter equation (CYBE)* is the equation for a linear map $R: \mathfrak{g} \to \mathfrak{g}$ of the form

$$[R(x), R(y)] = R([R(x), y] - [x, R^*(y)]), \; x, y \in \mathfrak{g}.$$

Theorem (Belavin-Drinfeld, 1982). Belavin-Drinfeld quadruples $\overline{\mathbf{\Gamma}} := (\Gamma_1, \Gamma_2, \gamma, R_0)$ parameterize the space of solutions of the CYBE.

$\overline{\mathbf{\Gamma}} := (\Gamma_1, \Gamma_2, \gamma, R_0)$ → Solution of CYBE R → A Poisson bracket $\{\,,\}_{\overline{\mathbf{\Gamma}}}$ on G

Example. Poisson brackets on $\mathrm{SL}_n(\mathbb{C})$:

$$\{f, g\}_{\overline{\mathbf{\Gamma}}}(X) := \langle R(\nabla_X f \cdot X), \nabla_X g \cdot X \rangle - \langle R(X \cdot \nabla_X f), X \cdot \nabla_X g \rangle$$

where $\nabla_X f := \left(\frac{\partial f}{\partial x_{ji}} \right)_{i,j=1}^{n}$; $< A, B > := \mathrm{trace}(\mathrm{AB})$.

Research program

More generally, one can associate a Poisson bracket
to a pair of BD quadruples $(\bar{\mathbf{\Gamma}}^r, \bar{\mathbf{\Gamma}}^c)$:

$$\{f, g\}_{(\bar{\mathbf{\Gamma}}^r, \bar{\mathbf{\Gamma}}^c)}(X) := \langle R^c(\nabla_X f \cdot X), \nabla_X g \cdot X\rangle - \langle R^r(X \cdot \nabla_X f), X \cdot \nabla_X g\rangle. \qquad (\mathrm{G} = \mathrm{SL}_n(\mathbb{C}))$$

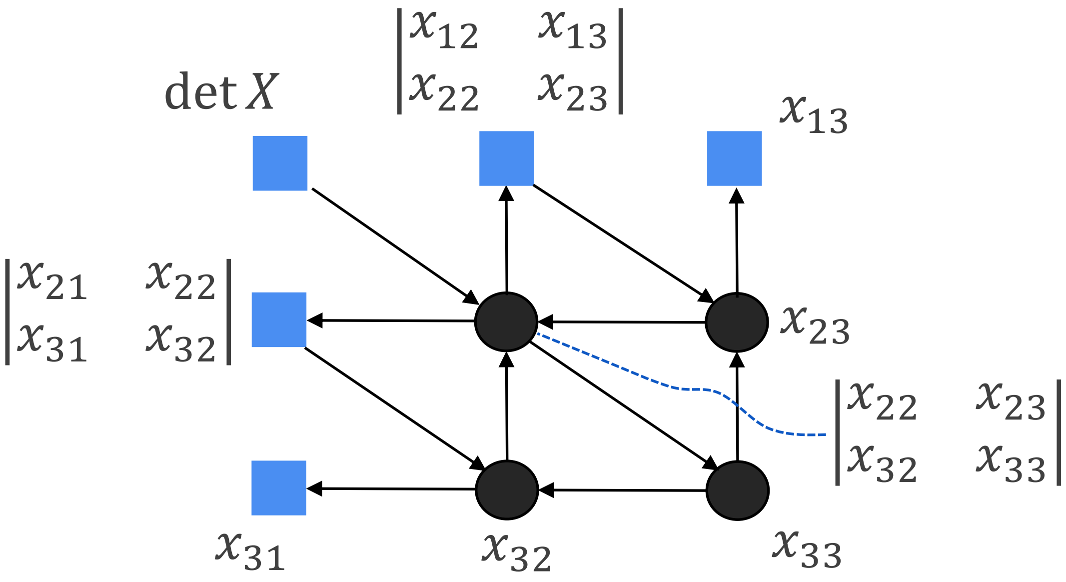
Conjecture (Gekhtman-Shapiro-Vainshtein, '11).
For any pair of BD quadruples $(\bar{\mathbf{\Gamma}}^r, \bar{\mathbf{\Gamma}}^c)$, there exists a (generalized) cluster structure $\mathcal{C}$ in $\mathbb{C}[G]$ such that

- $\mathcal{C}$ is compatible with $\{f, g\}_{(\bar{\mathbf{\Gamma}}^r, \bar{\mathbf{\Gamma}}^c)}$;
- $\bar{\mathcal{A}}_{\mathbb{C}}(\mathcal{C}) = \mathbb{C}[G]$.

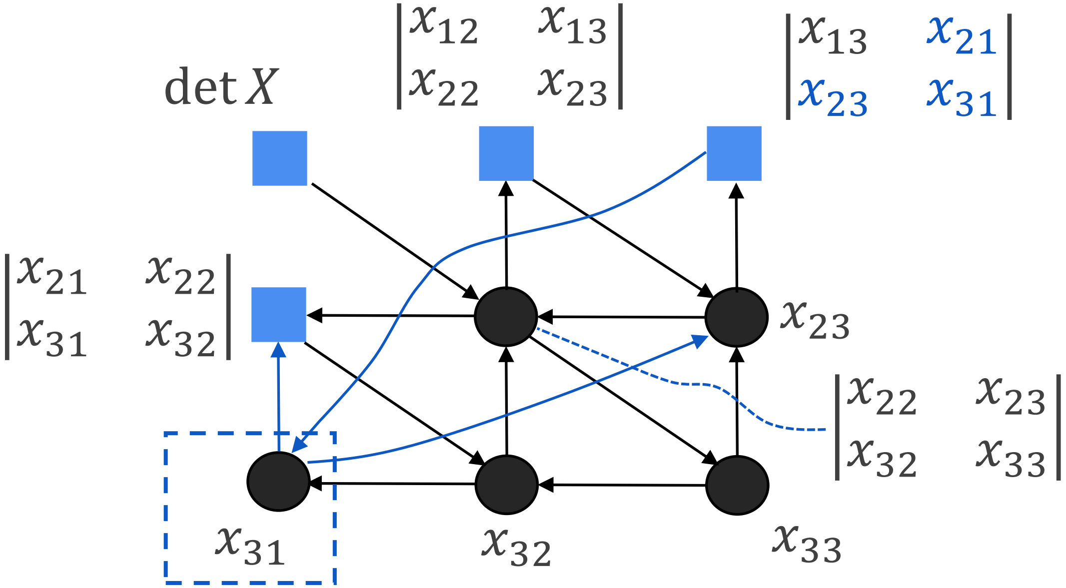
Remark.

- The conjecture extends to some other Poisson varieties (e.g., Drinfeld doubles and dual Poisson-Lie groups);
- $\mathcal{C}$ does not depend on R_0^r and R_0^c;
- The conjecture is solved for all *aperiodic* pairs $(\mathbf{\Gamma}^{\mathrm{r}}, \mathbf{\Gamma}^c)$.
  (aperiodic if $w_0\gamma_r w_0 \gamma_c^{-1}$ is nilpotent; beyond this case, one encounters generalized mutations, and that is the main hurdle to overcome)

An example in $\mathrm{GL}_3(\mathbb{C})$



Case $(\boldsymbol{\Gamma}_{\mathrm{st}}, \boldsymbol{\Gamma}_{\mathrm{st}})$



Case $(\boldsymbol{\Gamma}^{\mathrm{r}}, \boldsymbol{\Gamma}^{c})$

$\boldsymbol{\Gamma}^{\mathrm{r}} \coloneqq (\{2\}, \{1\}, 2 \mapsto 1),\ \boldsymbol{\Gamma}^{\mathrm{c}} \coloneqq \boldsymbol{\Gamma}_{\mathrm{st}}$

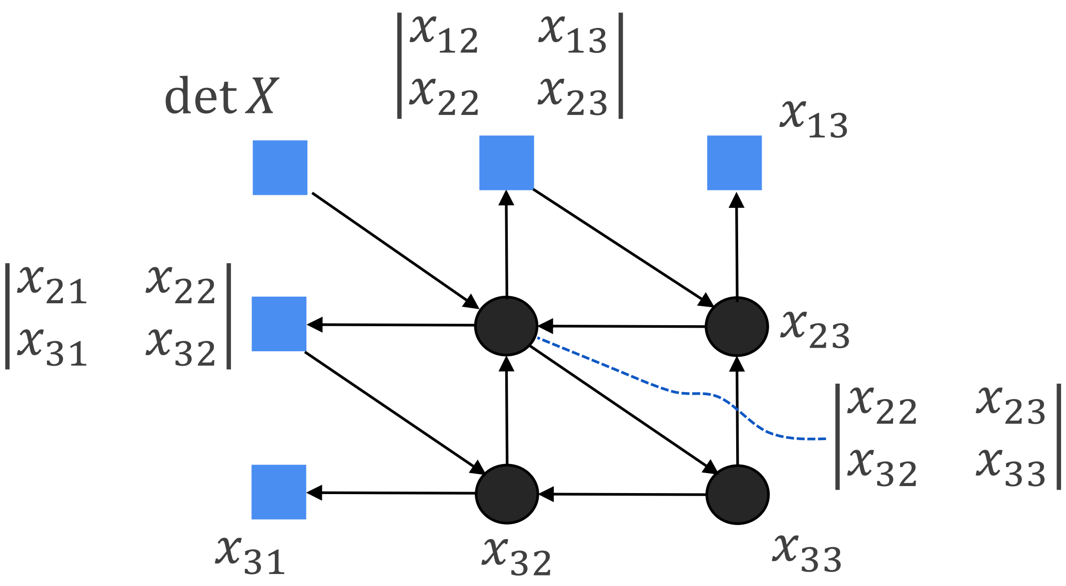
Crucial observation: There is a Poisson birational map

$$\mathcal{U}: (\mathrm{GL}_3(\mathbb{C}), \{\,,\}_{\mathrm{st}}) \dashrightarrow (\mathrm{GL}_3(\mathbb{C}), \{\,,\}_{(\boldsymbol{\Gamma}^{\mathrm{r}}, \boldsymbol{\Gamma}^{c})})$$

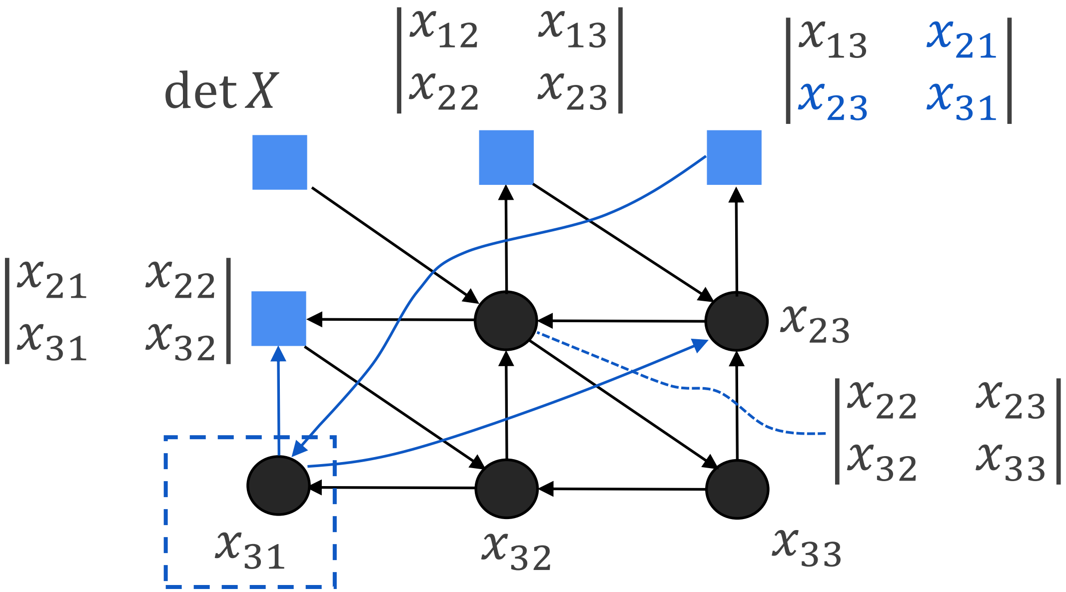
$$\mathcal{U}(X) = \begin{bmatrix} 1 & \frac{x_{21}}{x_{31}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot X$$

Call the variable x_{31} *marked*: it is frozen on the left, not frozen on the right, and it is in the denominator!

An example in $\mathrm{GL}_3(\mathbb{C})$



Case $(\mathbf{\Gamma}_{\mathrm{st}}, \mathbf{\Gamma}_{\mathrm{st}})$



Case $(\mathbf{\Gamma}^{\mathrm{r}}, \mathbf{\Gamma}^{c})$

$\mathbf{\Gamma}^{\mathrm{r}} := (\{2\}, \{1\}, 2 \mapsto 1),\ \mathbf{\Gamma}^{\mathrm{c}} := \mathbf{\Gamma}_{\mathrm{std}}$

The map $\mathcal{U}$ is also a *quasi-isomorphism* provided one does not mutate at the marked variable x_{31}.

For example,

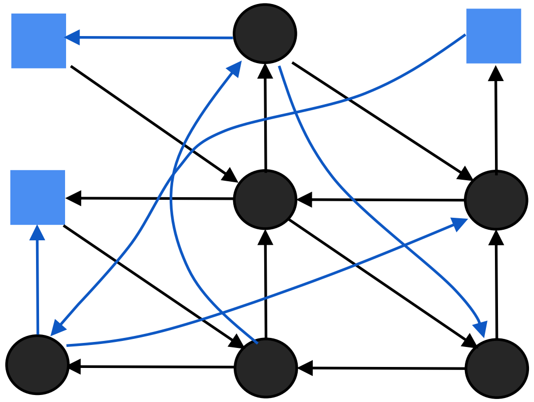
Mutation on the right

$$(\mathcal{U}^{-1})^*(x'_{23}) = x'_{23} \cdot x_{31}$$

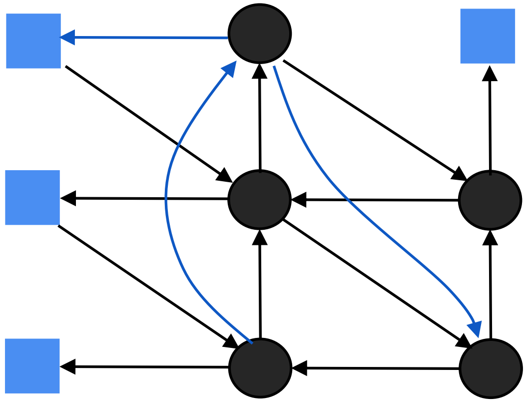
Mutation on the left

Always a monomial in the marked variables

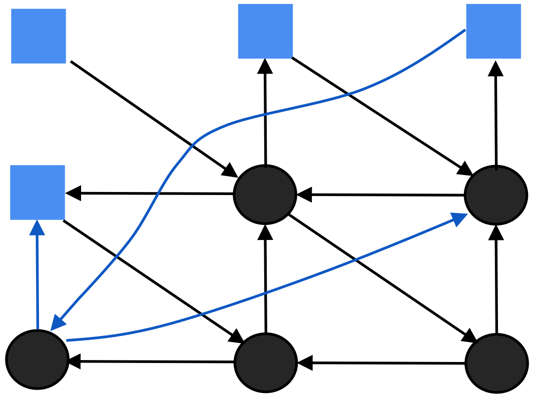
What other quivers look like for $\mathbf{GL}_3(\mathbb{C})$:



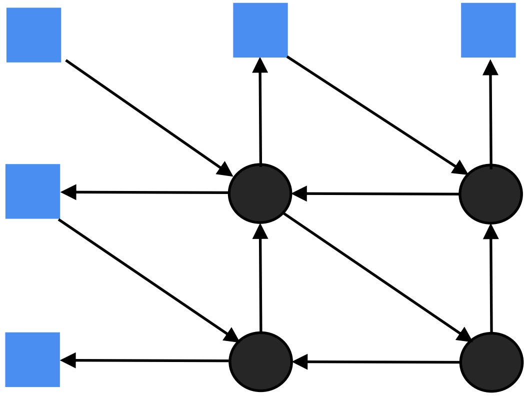
$\boldsymbol{\Gamma}^{\mathrm{r}} \coloneqq \boldsymbol{\Gamma}^{\mathrm{c}} \coloneqq (\{2\}, \{1\}, 2 \mapsto 1)$



$\boldsymbol{\Gamma}^{\mathrm{r}} \coloneqq \boldsymbol{\Gamma}_{\mathrm{st}},$
$\boldsymbol{\Gamma}^{\mathrm{c}} \coloneqq (\{2\}, \{1\}, 2 \mapsto 1)$



$\boldsymbol{\Gamma}^{\mathrm{r}} \coloneqq (\{2\}, \{1\}, 2 \mapsto 1),$
$\boldsymbol{\Gamma}^{\mathrm{c}} \coloneqq \boldsymbol{\Gamma}_{\mathrm{st}}$



$\boldsymbol{\Gamma}^{\mathrm{r}} \coloneqq \boldsymbol{\Gamma}^{\mathrm{c}} \coloneqq \boldsymbol{\Gamma}_{\mathrm{st}}$

Poisson rational quasi-isomorphism for $(G, \{\,,\}_{(\bar{\mathbf{\Gamma}}^r, \bar{\mathbf{\Gamma}}^c)})$

Gauss decomposition:

(unipotent radicals of opposite Borels)

For a generic $X \in G$, write $X = X_+ X_0 X_-$ where $X_\pm \in \mathcal{N}_\pm$ and $X_0 \in \mathcal{H}$.

(Cartan subgroup)

Define the maps $\rho_r: \mathcal{N}_+ \to \mathcal{N}_+$ and $\rho_c^*: \mathcal{N}_- \to \mathcal{N}_-$:

$$\rho_r(N_+) := \overleftarrow{\prod_{i \ge 1}} (\tilde{\gamma}_r)^i (N_+), \quad \rho_c^*(N_-) := \overrightarrow{\prod_{j \ge 1}} (\tilde{\gamma}_c^*)^j (N_-), \qquad N_\pm \in \mathcal{N}_\pm.$$

The map $\mathcal{U}: (G, \pi_{\mathrm{std}}) \dashrightarrow (G, \pi_{(\bar{\mathbf{\Gamma}}^r, \bar{\mathbf{\Gamma}}^c)})$:

$$\mathcal{U}(X) := \rho_r([XW_0]_+) \cdot X \cdot \rho_c^*([W_0 X]_-)$$

Theorem (GSV, ‘23). If (r_0^r, r_0^c) are the same for both $\pi_{(\bar{\mathbf{\Gamma}}^r, \bar{\mathbf{\Gamma}}^c)}$ and π_{std}, then $\mathcal{U}$ is Poisson.

Statement (proved ‘if’ part; unpublished): $\mathcal{U}^{-1}$ is rational if and only if $(\mathbf{\Gamma}^r, \mathbf{\Gamma}^c)$ is *aperiodic*.

(that is, $\gamma_r w_0 \gamma_c^* w_0^{-1}$ is nilpotent)

(Gekhtman-Shapiro-Vainshtein described $\mathcal{U}$ via Berenstein-Fomin-Zelevinsky parameters in their paper (2023); I found later that there is a closed formula)

Some open problems

1. Why are frozen variables frozen?

   A variable is frozen if and only if ______?______

   How to formalize ‘being frozen’?

2. (related) How to unfreeze frozen variables while preserving $\bar{\mathcal{A}}_{\mathbb{C}}(\mathcal{C}) = \mathbb{C}[G]$?

   (examples show that variables change as elements of $\mathbb{C}[G]$)

3. How to construct cluster structures (or some of their generalizations) on non-simply connected algebraic groups?

   (nothing is known even for $\mathrm{PSL}_n(\mathbb{C})$; how to even ask this question properly?)

4. How to quantize cluster structures with compatible with Belavin-Drinfeld brackets?

   (For example, there are cluster structures for so-called dual Poisson-Lie groups, and there are quantum cluster structures for quantum groups. The relation between dual Poisson-Lie groups and quantum groups is known; what about the relation between their cluster structures? And so on)

$\mathcal{A}$-$\mathcal{X}$ cluster duality and quantization

Setup. C – cluster structure, $\mathcal{F}$ its ambient field, $\{\ ,\ \}: \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ a Poisson bracket;

$(\boldsymbol{x}, B)$ – initial extended seed.

(M frozen variables, N cluster variables in each cluster)

Assume the compatibility condition:

$$B \cdot \Omega = [I \quad 0] \qquad (*)$$

Def. A y-variable is $y_k := \prod_{i=1}^{N+M} x_i^{b_{ki}}$

The compatibility condition $(*)$ is equivalent to

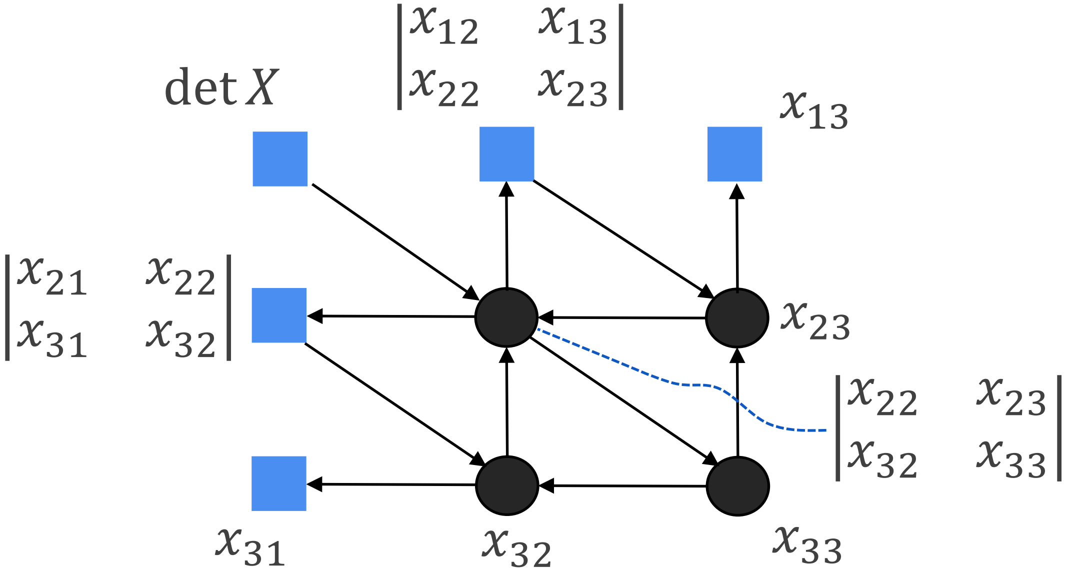
$$\{\log y_k, \log x_i\} = \delta_{ik}, \ \ i \in [1, N].$$

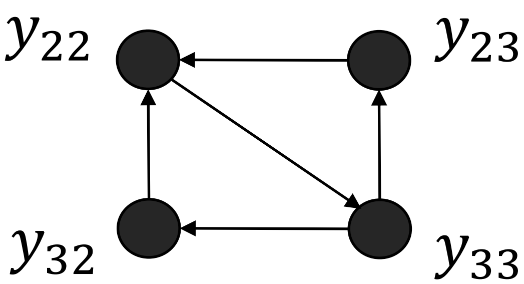
No matter what Poisson bracket we started with, if all extended clusters are log-canonical, the Poisson bracket between the y-variables is given by

$$\{\log y_i, \log y_j\} = \begin{cases} 1 & \text{if } x_j \to x_i \\ -1 & \text{if } x_i \to x_j \\ 0 & \text{no arrow between } x_i \text{ and } x_j \end{cases}$$

Conclusion. Poisson brackets between y-variables !

Example





$$y_{22} := \frac{\det\begin{vmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{vmatrix} \cdot \det\begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{vmatrix} \cdot x_{33}}{\det X \cdot x_{23} \cdot x_{32}}$$

$$y_{23} := \frac{\det\begin{vmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{vmatrix} \cdot x_{13}}{\det\begin{vmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{vmatrix} \cdot x_{33}} \qquad y_{32} := \frac{\det\begin{vmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{vmatrix} \cdot x_{31}}{\det\begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{vmatrix} \cdot x_{33}}$$

$$y_{33} := \frac{x_{23} \cdot x_{32}}{\det\begin{vmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{vmatrix}}$$

The y-variables have a different mutation pattern. For instance, if we mutate y_{33}, then the y-variables update as follows:

$$y'_{33} := y_{33}^{-1} \qquad y'_{22} := y_{22}(1 + y_{33})$$

$$y'_{23} := y_{23}(1 + y_{33}^{-1})$$

$$y'_{32} := y_{32}(1 + y_{33}^{-1})$$

This is compatible with the mutation of x_{33}.

That is, the new y-variables are the y-variables in the extended cluster obtained by mutation at x_{33}.

Easy to quantize:

$$\mathcal{O}_q := \mathbb{Z}[q^{\pm 1}][Y_1, Y_2, Y_3, Y_4 \mid Y_i Y_j = q^{\{\log y_i, \log y_j\}} Y_j Y_i]$$

This is an example of a χ-quantum cluster algebra.

Thank you