Lecture 2. Crystal bases for integrable highest weight modules

Reference

Bases cristallines des groupes quantiques (Kashiwara)

Recall The decomposition of $V(m) \otimes V(n) \curvearrowleft U_q(\mathfrak{sl}_2)$

has a nice behavior at $q = 0$

$$V(m) \otimes V(n) \ni w_l : \text{maximal vector of wt } l.$$

$$\bigoplus_i K f^{(i)} v_m \qquad \bigoplus_j K f^{(j)} v_n$$

$$f^{(k)} w_l \equiv f^{(i)} v_m \otimes f^{(j)} v_n$$

(i, j) is determined combinatorially.

The goal of this lecture is to introduce the notion of crystal base of $M \in \mathcal{O}_{int.}$ over $U_q(\mathfrak{g})$ $\mathfrak{g}$: symmetrizable Kac-Moody.

- <u>Kashiwara operator</u> (crystal operator)

We first need to define operators on M describing

i-string at $q=0$ for all $i \in I$.

$M \in \mathcal{O}_{int}$ $\quad i \in I$

Define $\tilde{e}_i, \tilde{f}_i : M \longrightarrow M$ as follows:

$v \in M_\lambda$.

This follows from M : semisimple over $U_q(sl_2)$

We have $v = \sum_{n \geq 0} f_i^{(n)} v_n$ for unique $v_n \in M_{\lambda - n\alpha_i}$: $e_i v_n = 0$.

Define $\tilde{e}_i v = \sum_{n \geq 1} f_i^{(n-1)} v_n$ $\quad$ $\tilde{f}_i v = \sum_{n \geq 0} f_i^{(n+1)} v_n$

<u>Rmk</u> These operators coincide with the ones for $U_q(sl_2)$

moving each vertex in $B(m) \otimes B(n)$ along " $\longrightarrow$ "

Def A crystal base of M is a pair $(\mathcal{L}, B)$

where $\mathcal{L}$: A_0-lattice of M satisfying

B : $\mathbb{k}$-basis of $\mathcal{L}/q\mathcal{L}$

① $\mathcal{L} = \bigoplus_{\lambda \in P} \mathcal{L}_\lambda \quad (\mathcal{L}_\lambda = M_\lambda \cap \mathcal{L})$ wt. space decomp.

② $B = \bigsqcup B_\lambda \quad (B_\lambda = B \cap \mathcal{L}_\lambda / q\mathcal{L}_\lambda)$

③ $\tilde{e}_i \mathcal{L} \subset \mathcal{L}, \quad \tilde{f}_i \mathcal{L} \subset \mathcal{L}$

invariant under crystal operators

④ $\tilde{e}_i B \subset B \cup \{0\}, \quad \tilde{f}_i B \subset B \cup \{0\}$

⑤ $\tilde{f}_i b = b' \iff b = \tilde{e}_i b' \quad (b, b' \in B)$

$\tilde{e}_i, \tilde{f}_i$: mutually inverses.

<u>Rmk</u> B has an I-colored oriented graph str.

$$b \xrightarrow{i} b' \iff b' = \tilde{f}_i b \quad (\tilde{e}_i b' = b)$$

B called a crystal (graph) of M.

can be viewed as a basis of M at $q = 0$.

For example, $V(m) \curvearrowleft U_q(\mathfrak{gl}_2)$

$$L(m) = \bigoplus_{k=0}^{m} A_0 f^{(k)} v_m \qquad B(m) = \{ f^{(k)} v_m \pmod{qL(m)} \}$$

crystal base

$$v_m \longrightarrow f v_m \longrightarrow \cdots \longrightarrow f^{(m)} v_m$$

The most important property of crystal base is the following.

<u>Theorem</u> (Tensor product theorem. Kashiwara 91)

$M_1, M_2 \in \mathcal{O}_{int}$ with a crystal base $(\mathcal{L}_i, B_i)$ $(i=1,2)$

$\Rightarrow$ $(\mathcal{L}_1 \otimes \mathcal{L}_2, B_1 \otimes B_2)$: a crystal base of $M_1 \otimes M_2$. where

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} (\tilde{f}_i b_1) \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2) \end{cases}$$

for $b_1 \otimes b_2 \in B_1 \otimes B_2$ and $i \in I$.

Recall

$$\overbrace{\bullet \longrightarrow \bullet \longrightarrow \cdots \underset{\tilde{e}_i b}{\bullet} \longrightarrow}^{\varepsilon_i(b)} \underset{b}{\bullet} \overbrace{\longrightarrow \underset{\tilde{f}_i b}{\bullet} \cdots \longrightarrow \bullet \longrightarrow}^{\varphi_i(b)} \bullet$$

$$\varepsilon_i(b) = \max\{ k \mid \tilde{e}_i^{\,k} b \neq 0 \} \qquad \varphi_i(b) = \max\{ k \mid \tilde{f}_i^{\,k} b \neq 0 \}$$

$$\begin{tikzcd}
 & & & 2 & 1 & 0 \\
\otimes & & & b_2 \arrow[r] & \tilde{f} b_2 \arrow[r] & \tilde{f}^2 b_2 \\
0 & b_1 \arrow[d] & & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[d] \\
1 & \tilde{f} b_1 & & \bullet \arrow[r] & \bullet & \bullet
\end{tikzcd}$$

Example

$$U_q(\mathfrak{gl}_n) = \langle e_i, f_i, q^{\pm \delta_j^\vee} \mid 1 \leq i \leq n-1,\ 1 \leq j \leq n \rangle$$

(1) $V = \bigoplus_{i=1}^{n} K v_i$: the natural repn

$$\begin{array}{ccc} v_i & \underset{e_i}{\overset{f_i}{\rightleftarrows}} & v_{i+1} \\ \delta_i & & \delta_{i+1} \end{array}$$

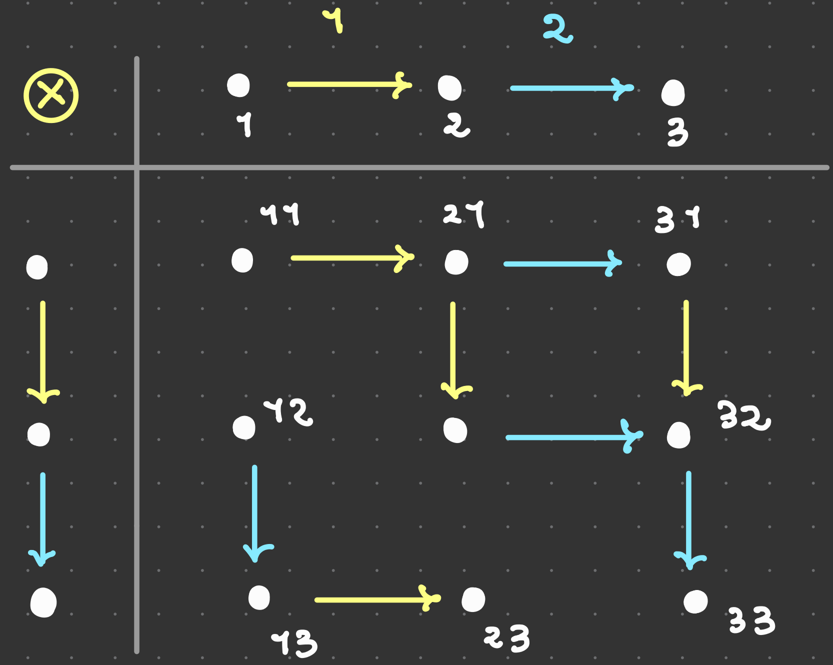
$\mathcal{L} := \bigoplus A_0 v_i \qquad B := \{ \overline{v_i} \ (\mathrm{mod}\ q\mathcal{L}) \}$: crystal base of V

In this case $\tilde{e}_i = e_i$, $\tilde{f}_i = f_i$

$$v_1 \xrightarrow{1} v_2 \xrightarrow{2} v_3 \longrightarrow \cdots \xrightarrow{n-1} v_n \qquad \xrightarrow{i} = \tilde{f}_i$$

$V^{\otimes 2}$ has a crystal base $(\mathcal{L}^{\otimes 2}, B^{\otimes 2})$

$n = 3$



$(ij = v_i \otimes v_j)$

Recall $V \otimes V \cong \underset{\substack{\| \\ 2\varpi_1}}{V(2\delta_1)} \oplus \underset{\substack{\| \\ \varpi_2}}{V(\delta_1 + \delta_2)}$ (e.g. by Pieri rule.)

How are these two decompositions are related?

② $$\underline{W^{\epsilon} = \bigoplus_{m \in X_\epsilon} K\, m} \qquad \epsilon = 0, 1 \qquad X_\epsilon = \begin{cases} \mathbb{Z}_{\geq 0}^n & (\epsilon = 0) \\ \mathbb{Z}_2^n & (\epsilon = 1). \end{cases}$$

W^ϵ : a $U_q(\mathfrak{gl}_n)$-module

$$e_i\, m = [m_{i+1}]\,(\cdots, m_i + 1, m_{i+1} - 1, \cdots)$$

$$f_i\, m = [m_i]\,(\cdots\ m_i - 1, m_{i+1} + 1, \cdots)$$

$$q^{\delta_i^\vee}\, m = q^{m_i}\, m$$

$$\underline{W_k^\epsilon := \bigoplus_{\sum m_i = k} K\, m} \quad \text{submodule} \quad \cong \begin{cases} V(k\varpi_1) & \epsilon = 0 \\ V(\varpi_k) & \epsilon = 1. \end{cases}$$

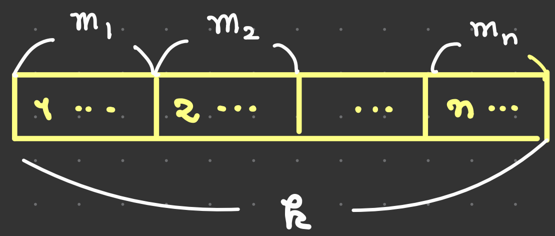
$$\mathcal{L}_k^{\epsilon} := \bigoplus A_0 \frac{1}{\prod [m_i]!} \mathbf{m} \qquad B_k^{\epsilon} = \left\{ \frac{1}{\prod [m_i]!} \overline{\mathbf{m}} \pmod{q \mathcal{L}_k^{\epsilon}} \right\}$$

: crystal base of W_k^{ϵ}

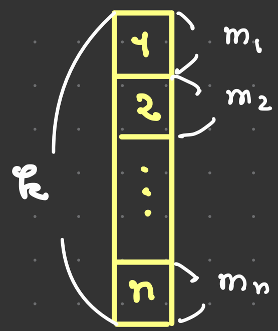
$$(\cdots, m_i, 0, \cdots) \xrightarrow{f_i^{(k)}} (\cdots, m_i - k, k, \cdots)$$

$U_q(\mathfrak{gl}_2)$-h.w. vector

Identify $\mathbf{m}$ with



$\epsilon = 0$



$\epsilon = 1$

$n = 3$

$B(2\varpi_1)$

$$\begin{tikzcd}
11 \arrow[r] & 12 \arrow[r] \arrow[d] & 13 \arrow[d] \\
 & 22 \arrow[r] & 23 \arrow[d] \\
 & & 33
\end{tikzcd}$$

$B(\varpi_2)$

$$\begin{tikzcd}
\begin{smallmatrix}1\\2\end{smallmatrix} \arrow[d] & \\
\begin{smallmatrix}1\\3\end{smallmatrix} \arrow[r] & \begin{smallmatrix}2\\3\end{smallmatrix}
\end{tikzcd}$$

$$B^{\otimes 2} \cong B(2\varpi_1) \sqcup B(\varpi_2)$$

In the previous example, we have seen that

decomposition of a crystal = decomposition of a module

Theorem A (Existence of a crystal base. Kashiwara 91)

$\lambda \in P_+$

$V(\lambda)$ has a crystal base $(\mathcal{L}(\lambda), B(\lambda))$ where

$$\mathcal{L}(\lambda) = \sum_{i_1, \cdots, i_r} A_0 \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} v_\lambda$$

$$B(\lambda) = \{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} v_\lambda \pmod{q\mathcal{L}(\lambda)} \} \setminus \{0\}$$

Theorem B (Uniqueness of a crystal base, Kashiwara 91)

$M \in \mathcal{O}_{int.}$ with a crystal base $(\mathcal{L}, B)$

Then $\exists\, \varphi : M \xrightarrow{\cong} \bigoplus_{P_+} V(\lambda)^{\oplus m_\lambda}$ such that

$$\varphi|_{\mathcal{L}} : \mathcal{L} \longrightarrow \bigoplus \mathcal{L}(\lambda)^{\oplus m_\lambda}$$

$$\bar{\varphi}|_B : B \longrightarrow \bigsqcup B(\lambda)^{\oplus m_\lambda}$$

where $\bar{\varphi} : \mathcal{L}/q\mathcal{L} \longrightarrow \bigoplus \left(\mathcal{L}(\lambda)/q\mathcal{L}(\lambda) \right)^{\oplus m_\lambda}$

Rmk

① By Existence thm + Complete reducibility of $\mathcal{O}_{\text{int}}$

any $M \in \mathcal{O}_{\text{int}}$ has a unique crystal base

② $M \in \mathcal{O}_{\text{int}}$ with a crystal base $(\mathcal{L}, B)$

By Uniqueness of crystals,

$$B \cong \bigsqcup B(\lambda)^{\oplus m_\lambda} \qquad m_\lambda = \mathrm{Hom}_{U_q(\mathfrak{g})}(V(\lambda), M)$$

decomp of M into $V(\lambda)$'s $\Longleftrightarrow$ decomp. of B into $B(\lambda)$'s.

③ By construction,

$$B(\lambda) = \{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} v_\lambda \ (\mathrm{mod}\ q\mathcal{L}(\lambda)) \} \setminus \{0\} \ : \text{connected.}$$

$b \in B(\lambda)$ $\quad \tilde{e}_i b = 0$ for all i $\iff$ $b = v_\lambda$

decomp. of B into $B(\lambda)$'s.

$\iff$ finding all $b \in B$ such that $\tilde{e}_i b = 0$ for all i.

In fact. the connected component of b under $\tilde{f}_i$'s

$$\cong B(\lambda) \quad \text{where} \quad \lambda = \mathrm{wt}(b).$$

Tensor product decomposition.

$\mu, \nu \in P_+$

$$V(\mu) \otimes V(\nu) = \bigoplus_{\lambda \in P_+} V(\lambda)^{\oplus c_{\mu\nu}^{\lambda}}$$

$$B(\mu) \otimes B(\nu) = \coprod_{\lambda \in P_+} B(\lambda)^{\oplus c_{\mu\nu}^{\lambda}}$$

$c_{\mu\nu}^{\lambda}$ = # of $b_1 \otimes b_2 \in B(\mu) \otimes B(\nu)$ such that

$$\tilde{e}_i(b_1 \otimes b_2) = 0 \quad \text{for all } i$$

$$\mathrm{wt}(b_1 \otimes b_2) = \lambda$$

Rmk

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2) \end{cases}$$

$$\tilde{e}_i b_* = 0 \iff \varepsilon_i(b_*) = 0$$

$$\tilde{e}_i(b_1 \otimes b_2) = 0 \quad (i \in I)$$

$\Rightarrow$ $\varphi_i(b_1) \geq \varepsilon_i(b_2)$ for all i + $\underline{\tilde{e}_i b_1 = 0 \text{ for all } i}$

$\Rightarrow$ $b_1 = v_\lambda$ & $\varepsilon_i(b_2) \leq \langle h_i, \lambda \rangle$

Converse also holds (hence equivalent condition)

Therefore,

$$B(\mu) \otimes B(\nu) = \coprod_{\substack{b_2 \in B(\nu) \\ \varepsilon_i(b) \leq \langle h_i, \lambda \rangle}} B(\mu + \mathrm{wt}(b_2))$$

Once we have a good realization of $B(\nu)$, we have a formula

$$c^{\lambda}_{\mu\nu} = \# \left\{ b \in B(\nu) \;\middle|\; \begin{array}{l} \varepsilon_i(b) \leq \langle h_i, \mu \rangle \quad (i \in I) \\ \mu + \mathrm{wt}(b) = \lambda \end{array} \right\}$$

Rmk $c^{\lambda}_{\mu\nu}$: Littlewood-Richardson coeff. when $\mathfrak{g} = \mathfrak{gl}_n \ (\mathfrak{sl}_n)$

$\exists$ a combinatorial formula given as # of LR tableaux of shape λ/μ content ν

Example

① $B^{\otimes 2}$, $\quad \mu = \varpi_1 \qquad (\langle h_i, \mu \rangle)_I = (1, 0, \cdots, 0)$

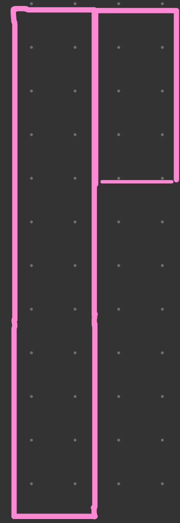
$v_1 \otimes \underbrace{v_1}_{0}$, $v_1 \otimes \underbrace{v_2}_{1}$: the only maximal vectors in $B^{\otimes 2}$

$$B^{\otimes 2} = B(\varpi_1) \otimes B(\varpi_1) = B(\varpi_1 + \varpi_1) \amalg B(\varpi_2)$$

② $B(\varpi_a) \otimes B(\varpi_b)$

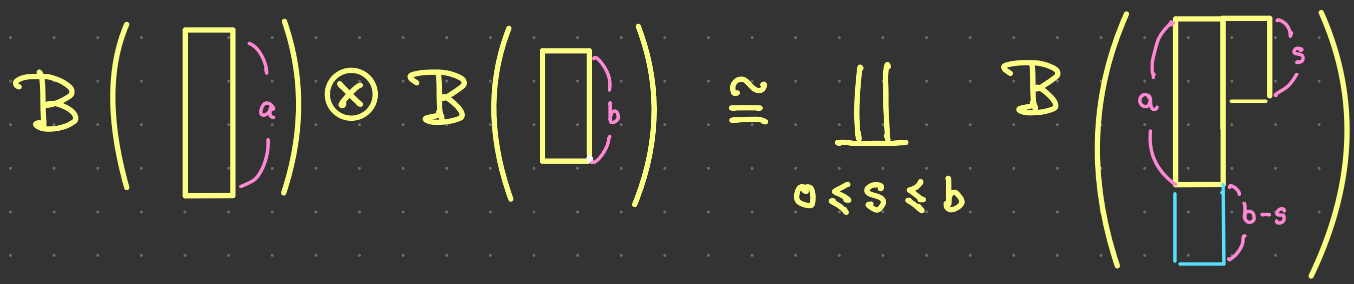
a maximal vector $= \begin{array}{|c|} \hline 1 \\ 2 \\ \vdots \\ a \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 1 \\ \vdots \\ s \\ \hline a+1 \\ a+2 \\ \vdots \\ \hline \end{array} \qquad (0 \leq s \leq \min(a, b))$

$$\mathrm{wt}\left(\begin{array}{|c|}\hline 1\\ 2\\ \vdots\\ a\\ \hline\end{array}\otimes\begin{array}{|c|}\hline 1\\ \vdots\\ s\\ \hline a+1\\ a+2\\ \vdots\\ \hline\end{array}\right) = \mathrm{wt}\left(\begin{array}{|c|}\hline 1\\ 2\\ \vdots\\ a\\ \hline a+1\\ a+2\\ \vdots\\ \hline\end{array}\right) + \mathrm{wt}\left(\begin{array}{|c|}\hline 1\\ \vdots\\ s\\ \hline\end{array}\right)$$



two-column Young diagram (or partition)

Then we recover the Pieri's rule



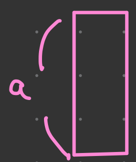
$(a \geq b)$

③ $\lambda \in P_+$ $\qquad \lambda = \sum_{i=1}^{m} m_i \delta_i \qquad (m_i \in \mathbb{Z},\ m_1 \geq \cdots \geq m_n)$

We may assume $(m_1 \geq \cdots \geq m_n \geq 0) \longleftrightarrow$ a partion (or its Young diagram)

More generally, we can prove (by the same arguments) that

$$B(\mu) \otimes B((1^a)) \cong \coprod_{\lambda \in P_+} B(\lambda)$$



λ/μ : a vertical strip of length a