A LAX OPERATOR REALIZATION OF CLASSICAL $\mathcal{W}$-ALGEBRAS

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Goal:

- give some intuition on what classical W-algebras are.
- explain my joint work with U. Suh on their supersymmetric (SUSY) counterparts.

Plan:

- intro
- nonSUSY classical W-algebras
- SUSY classical W-algebras

Keywords: differential algebra, Lie algebra, integrable system.

What are W-algebras?

	Classical	Quantum
Finite	CLASSICAL MECHANICS POISSON ALGEBRA	QUANTUM MECHANICS ASSOCIATIVE ALGEBRA
Infinite	CLASSICAL FIELD THEORY POISSON VERTEX ALGEBRA ($\Leftrightarrow$ HAMILTONIAN DIFF. OPERATOR)	QUANTUM FIELD THEORY VERTEX ALGEBRA

Poisson algebra:

- $S(\mathfrak{g})$, $\{\cdot,\cdot\}$

$$(*)\quad \{f,g\} = \sum_{i,j=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}[x_i, x_j]$$

Associative algebra:

- $U(\mathfrak{g})$
- $\mathrm{gr}\, U(\mathfrak{g}) = S(\mathfrak{g})$

semi-simple Lie algebra $\mathfrak{g}$

Poisson Vertex algebra:

- $\mathcal{V}^k(\mathfrak{g}) = \mathrm{gr}\, V^k(\mathfrak{g})$
- alg. of differential polynomials.

$\widetilde{(*)}$: " matrix $[x_i, x_j]_{1\leq i,j\leq n}$ "

$\Rightarrow$ matrix diff. operator

Vertex algebra:

- Universal affine $V^k(\mathfrak{g})$

$= \{$ vector space

$+$ fields $\in \mathrm{End}\, V^k(\mathfrak{g})[[z, z^{-1}]]$

satisfying some axioms $\}$

$S(\mathfrak{g}, 0) = S(\mathfrak{g})$

$S(\mathfrak{g}, f)$ (Slodowy, 80)

$S(\mathfrak{g}, f^{pr}) = S(\mathfrak{g})^G$

$U(\mathfrak{g}, 0) = U(\mathfrak{g})$

$U(\mathfrak{g}, f)$ (Premet, 02)

$U(\mathfrak{g}, f^{pr}) \simeq Z(U(\mathfrak{g}))$ (Kostant, 78)

nilpotent element f in $\mathfrak{g}$

$W(\mathfrak{g}, f^{pr})$ (Drinfeld, Sokolov 85)

$\vdots$

$W(\mathfrak{g}, f)$ (De Sole - Kac - Valeri 13)

$W(\mathfrak{g}, 0) = V(\mathfrak{g})$

- $W(\mathfrak{sl}_2, f^{pr}) \simeq$ Virasoro; $W(\mathfrak{sl}_3, f^{pr})$ $\mathcal{W}_3$ (Zamolodchikov 85)
- $W(\mathfrak{g}, f^{pr})$ (Feigin - Frenkel 90)
- $W(\mathfrak{g}, f)$ (Kac - Wakimoto 04)

Long term goals:

structure theorems for $W^k(\mathfrak{g}, f)$ and $\mathcal{W}^k(\mathfrak{g}, f)$

Applications:

integrable systems

representation theory

Today's focus:

classical W-algebras $\nearrow$ non-supersymmetric (De Sole Kac Valeri)

$\searrow$ SUSY (C. Suh)

$\underline{S(\mathfrak{gl}_N, f^{pr}) = S(\mathfrak{gl}_N)^{GL_N}}$ is generated by

$$\det \begin{vmatrix} e_{11}+z & e_{21} & e_{31} & \cdots & e_{N1} \\ e_{12} & e_{22}+z & e_{32} & \cdots & e_{N2} \\ e_{13} & e_{23} & e_{33}+z & \cdots & e_{N3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ e_{1N} & e_{2N} & e_{3N} & \cdots & e_{NN}+z \end{vmatrix} =$$

$$z^N + s_1 z^{N-1} + s_2 z^{N-2} + \cdots + s_N$$

$\underline{Z(U(\mathfrak{gl}_N)) \simeq U(\mathfrak{gl}_N, f^{pr})}$ is generated by

row det of

$$
\begin{vmatrix}
E_{11}+z & E_{21} & E_{31} & \cdots & E_{N1} \\
E_{12} & E_{22}+z-1 & E_{32} & \cdots & E_{N2} \\
E_{13} & E_{23} & E_{33}+z-2 & \cdots & E_{N3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
E_{1N} & E_{2N} & E_{3N} & \cdots & E_{NN}+z-N+1
\end{vmatrix}
$$

- what about classical W-algebras?

CLASSICAL W-ALGEBRAS

NON SUPERSYMMETRIC CASE

Lax operators, origins [Lax 68]

$$u_t = u''' + 3uu' \qquad u = u(x,t)$$

Korteweg - de Vries equation

$$\Longleftrightarrow \quad L_t = [M, L]$$

$$L = \partial^2 + u$$

$$M = \partial^3 + \alpha u \partial + \beta u'$$

$\Rightarrow$ KdV is a integrable system.

Suppose. L is a differential operator.

$\mathcal{V}$ is the algebra of diff. polynomials generated by L.

Definition A evolutionary derivation $\partial_t : \mathcal{V} \to \mathcal{V}$ is a derivation such that $[\partial_t, \partial] = 0$.

Question. For which L $\exists$ $\Pi \in \mathcal{V}[\partial]$ such that

$$\partial_t(L) = [\Pi, L] \text{ makes sense?}$$

Examples.

$L = \partial + u$ ✓ ✓ $L = \partial^3 + u\partial + v$

$L = \partial^2 + u$ ✓ ✗ $L = \partial^3 + u$

Gelfand-Dickey Hierarchies [GD78]

Lax operator

$$L = \partial^N + u_1 \partial^{N-1} + \dots + u_N$$

Differential algebra

$$\mathcal{W} = \mathbb{C}[u_i^{(k)},\ k \geq 0,\ 1 \leq i \leq N]$$

Family of evolutionary derivations

$$\partial_{t_k}(L) = [(L^{k/N})_+, L], \quad k = 1, 2, 3, \dots$$

pseudo-differential operators

$$\mathcal{W}[\partial] \subset \mathcal{W}((\partial^{-1})), \quad \partial^{-1} \cdot a = \sum_{n \geq 0} (-1)^n a^{(n)} \partial^{-n-1}$$

Properties of the GD hierarchies:

- $[\partial_{t_k}, \partial_{t_\ell}] = 0 \qquad k, \ell \geq 1$

- $\partial_{t_\ell}(\operatorname{res} L^{k/N}) \in \partial \mathcal{W} \qquad$ " "

- These derivations are Hamiltonian:

1) $\exists\, H \in \mathcal{M}_N(\mathcal{W}[\partial])$,

$$\{\textstyle\int f, \int g\} := \int \frac{\delta f}{\delta \vec{u}} \cdot H\left(\frac{\delta g}{\delta \vec{u}}\right)$$ Lie bracket on $\mathcal{W}/\partial\mathcal{W}$

2) $\int \operatorname{res} L^{k/N} \in \mathcal{W}/\partial\mathcal{W} \xrightarrow[\frac{\delta}{\delta \vec{u}}]{} \mathcal{W}^N \xrightarrow[H]{} \mathcal{W}^{\cdot N} \Rightarrow \partial_{t_k}(L)$.

Gelfand - Dickey Hamiltonian Operator

$$\{\textstyle\int f, \int g\} := \int \mathrm{res}\left[\left(L \frac{\delta f}{\delta L}\right)_+ L \frac{\delta g}{\delta L} - L\left(\frac{\delta f}{\delta L} L\right) \frac{\delta g}{\delta L}\right]$$

defines a Lie algebra bracket on $\mathcal{W}/\partial\mathcal{W}$

called the quadratic GD bracket

$\Longleftrightarrow$ Hamiltonian operator $H \in \mathcal{M}_N(\mathcal{W}[\partial])$

Where are Lie algebras ? [DS 85]

- matrix Lax operator $\mathcal{L} = \partial + q + f + zs,$

$$\begin{cases} \mathfrak{g} = \mathfrak{gl}_N \\ q \in \mathfrak{b}_+ \end{cases} \qquad f = \sum_{i=2}^{N} e_{i+1\, i} \qquad s = e_{1N} \qquad \mathcal{V} := \langle q, \partial \rangle$$

- ($f + zs$ semi-simple $\Rightarrow$)

DS construct $\{ \partial_{\tilde{t}_n} : \mathcal{V} \longrightarrow \mathcal{V} \}_{n \geq 1}$

1) $[\partial_{\tilde{t}_k}, \partial_{\tilde{t}_n}] = 0 \qquad k, n \geq 1$

2) Hamiltonian for the affine Poisson Vertex Algebra bracket

$$[a_\lambda b] = [a, b] + \lambda (a | b) \qquad a, b \in \mathfrak{g}.$$

GD ⊂ DS

- $\exists!$ $\mathfrak{n}_+$-valued $N(q)$ s.t. $e^{\operatorname{ad} N} \cdot \mathcal{L} =$

$$\begin{pmatrix} \partial & 0 & & w_1(q) \\ 1 & \partial & & w_2(q) \\ 0 & 1 & \cdots & w_3(q) \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & \partial + w_N(q) \end{pmatrix}$$

- <u>Theorem</u> [DS 85]

1) $\tilde{\partial}_{t_n} \big|_{\langle w_i(q), \partial \rangle} = \partial_{t_n}$

2) affine bracket $\big|_{\langle w_i(q), \partial \rangle}$ = GD bracket

matrix Lax operator $\mathcal{L}$ $\xRightarrow[\text{gauge fixing}]{}$ scalar Lax operator L

Classical $W(\mathfrak{g},f)$: modern definition [DSKV 13]

Data: $\mathfrak{g}$ semi-simple, f nilpotent.

$$\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k\ , \quad \mathfrak{g}_k \text{ eigenspace of } \operatorname{ad} h\ , \quad \{h,e,f\} \simeq \mathfrak{sl}_2 .$$

affine PVA:

$$V(\mathfrak{g}) := \mathbb{C}\langle \mathfrak{g}, \partial \rangle, \text{ with } \lambda\text{-bracket}$$

$$[a_\lambda b] = [a,b] + \lambda (a|b), \quad a,b \in \mathfrak{g}.$$

$W(\mathfrak{g},f)$:

$$\left(V(\mathfrak{g}) \Big/ \langle m - (f|m) \rangle_{m \in \mathfrak{g}^{\geq 2}} \right)^{\operatorname{ad}_\lambda \mathfrak{g}^{\geq 1}}$$

Structure Theorem [DSKV17]

- $W(\mathfrak{g}, f)$ is generated by the coefficients of a (rational) matrix differential operator $\mathcal{L}(\mathfrak{g}, f)$

- There is a universal formula for $\{\mathcal{L}(z)_\lambda \mathcal{L}(w)\}$ for each type called Adler identity.

- $\mathcal{L}(\mathfrak{g}, f)$ = some quasi-determinant of $\mathcal{L}(\mathfrak{g}, 0)$

- They construct integrable system of Lax form for each $W(\mathfrak{g}, f)$.

type A classical W-alg.

$N = \underbrace{p_1 + p_1 + \dots + p_1}_{r_1 \text{ times}} + p_2 + \cdots$

① $\mathcal{L}(\mathfrak{g}, f) = \rho \begin{pmatrix} \partial + e_{11} & e_{21} & e_{31} & & \\ e_{12} & \partial + e_{22} & e_{32} & & \\ e_{13} & e_{23} & \partial + e_{33} & & \\ & & & \ddots & \\ & & & & \partial + e_{NN} \end{pmatrix}_{I_1, J_1}$,

- ρ projects e_{ij}'s mod the diff. ideal $\langle m - (f|m), m \in \mathfrak{m} \rangle$
- $A_{I_1, J_1} = (I_1 A^{-1} J_1)^{-1}$, I_1 and J_1 constant matrices of size $r_1 \times N$, $N \times r_1$.

② $\{ \mathcal{L}_{ij}(z)_\lambda \mathcal{L}_{hk}(w) \} = \mathcal{L}_{hj}(w + \lambda + \partial)(z - w - \lambda - \partial)^{-1} \mathcal{L}_{ik}^*(\lambda - z)$
$\qquad - \mathcal{L}_{hj}(z)(z - w - \lambda - \partial)^{-1} \mathcal{L}_{ik}(w)$

(Adler identity)

classical W-alg	Lax operators
• sl_N, f^{pr}	• $\partial^N + u_1 \partial^{N-2} + \dots + u_{N-1}$
• $so(2N+1)$, f^{pr}	• $\partial^{2N+1} + \partial^N u_1 \partial^{N-1} + \dots + \partial u_N$
• $sp(2N)$, f^{pr}	• $\partial^{2N} + \partial^{N-1} u_1 \partial^{N-1} + \dots + u_N$
• $so(2N)$, f^{pr}	• $\partial^{2N-1} + \partial^{2N-3} u_1 + u_1 \partial^{2N-3} + \dots + u_N \partial^{-1} u_N$
• E_6, E_7, E_8, F_4, G_2, f	???
X	• $\partial^4 + \partial a \partial + b$, $\begin{cases} b = \frac{1}{2} a'' + \frac{1}{4} a^2 - \frac{c}{3} u \\ a = -\frac{u'''}{u'} + \frac{1}{2}\left(\frac{u''}{u'}\right)^2 - \frac{\alpha(u)}{u'^2} \end{cases}$

SUPERSYMMETRIC

CLASSICAL W-ALGEBRAS

Supersymmetry

- "bosons and fermions come in pairs"
- grassmann coordinates :
  - x even $\qquad \theta_1, \dots, \theta_N$ odd

$$\theta_i \theta_j + \theta_j \theta_i = 0$$

- $N=1$ SUSY: $\quad D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial x} \Rightarrow D^2 = \partial$

- SUSY algebra of differential polynomials
  - $\mathbb{C}[u_i^{(k)} \mid 1 \leq i \leq M, \ k \geq 0]$
  - u_i even or odd
  - $u_i^{(k)} = D^k(u_i)$

$N=1$ SUSY $\mathcal{W}(\mathfrak{g}, f)$

- As quantum objects: Madsen - Ragoucy '94.
  (SUSY BRST complexes)
- foundations of SUSY Vertex Algebras:
  Heluani - Kac '07
- Molev - Ragoucy - Suh '19 showed that [MR94] algebras fit in [HK07] language.
- As classical objects: Suh '20
  Ragoucy - Song - Suh '21

classical $\mathcal{W}(\mathfrak{g}, f)$, definition:

- data. - $\mathfrak{g}$ simple Lie superalgebra,
  - f odd nilpotent in $\mathfrak{g}$ such that,
  - $f \in osp(1|2) \subset \mathfrak{g}$.

- example. if $\mathfrak{g}$ of type sl or osp and f principal,

$$\mathfrak{g} = sl(n \pm 1 \mid n)$$

$$\mathfrak{g} = osp(2n, 2n \pm 1, 2n+2 \mid 2n)$$

Affine SUSY PVA

- SUSY differential algebra generated by $\mathfrak{g}$ and odd derivation D : $\mathcal{V}(\mathfrak{g})$.

- SUSY PVA bracket:

$$[a_\chi b] = [a,b] + \chi(a|b), \quad a,b \in \mathfrak{g}.$$

- $W(\mathfrak{g}, f)$ constructed from $\mathcal{V}(\mathfrak{g})$ via Hamiltonian reduction based on the grading induced by $(\mathrm{ad}\, F)$.

Problem

Can we realize these algebras as
Gelfand-Dickey algebras ?

- generators of $W(\mathfrak{g}, f)$ = coeff. of $\mathcal{L}(\mathfrak{g}, f)$
- "universal identity" for the χ-bracket

$$\{\mathcal{L}(z)_\chi \mathcal{L}(w)\} =$$

- construct SUSY integrable systems.

[CS24]: Answer for $\mathfrak{gl}(n+1 \mid n)$ f principal.

- $N \geqslant 2$.
- $L = D^N + u_1 D^{N-1} + \cdots + u_N \qquad p(u_i) \equiv i \; [2]$
- $\mathcal{W}_N = \mathbb{C}\left[u_i^{(k)} \mid 1 \leqslant i \leqslant N, \; k \geqslant 0 \right]$
- hierarchy of compatible equations on $\mathcal{W}_N$:

$$\partial_\ell (L) = \left[\left(L^{2\ell/N} \right)_+ , L \right]$$

$$[\partial_\ell, D] = 0$$

$$[\partial_\ell, \partial_{\ell'}] = 0$$

parity of N matters

N = 2n: $\left(\operatorname{res} L^{q/n}\right)_{q \geq 1}$ are conserved.

$\left(\partial_\ell(h) \in DW_N\right)$. These are odd

N = 2n+1: $\left(\operatorname{res} L^{\frac{2q-1}{2n+1}}\right)_{q \geq 1}$ are conserved.

These are even.

derivations $(\partial_\ell)_{\ell \geq 1}$ $\leftarrow$ Hamiltonian operator? $\rightarrow$ conserved densities

Hamiltonian operator in mathematical physics

- variational derivative $\mathcal{W}_N / D\mathcal{W}_N \to (\mathcal{W}_N)^N$

- evolutionary derivation $X_F : \mathcal{W}_N \to \mathcal{W}_N$,

$$[X_F, D] = 0, \qquad F \in (\mathcal{W}_N)^N$$

$$X_F(u_i) = F_i$$

- They form a Lie superalgebra.

- $H \in \mathcal{M}_N(\mathcal{W}_N[D])$ is Hamiltonian $\Leftrightarrow$

$$\left\{ X_{H\left(\frac{\delta f}{\delta L}\right)},\ f \in \mathcal{W}_N / D\mathcal{W}_N \right\} \text{ sub Lie super algebra.}$$

Theorem 1.

$m = n$ or $n+1$, $N = m+n$, f principal, $\mathfrak{g} = \mathfrak{gl}(m|n)$.

$\exists\ \phi : \mathcal{W}(\mathfrak{g}, f) \longrightarrow W_N$,

differential algebra isomorphism

$$\int \phi \{ \phi^{-1}(a)_x \, \phi^{-1}(b) \} \Big|_{x=0} =$$

$$(-1)^{N+a} \int \operatorname{res} \left[\left(L \frac{\delta a}{\delta L} \right)_+ L \frac{\delta b}{\delta L} - L \left(\frac{\delta a}{\delta L} L \right)_+ \frac{\delta b}{\delta L} \right]$$

for all $a, b \in W_N$.

Remarks.

- $N=1$ SUSY PVA bracket

  $\Longleftrightarrow$ odd Hamiltonian operator

- The bracket on W_N / DW_N is known as SUSY Gelfand-Dickey bracket, but there are no proof of the Jacobi identity.

- $$\{L(z)_\chi L(w)\} = \langle L(D+\chi+w)(D+\chi+w-z)(D^2-\chi^2+w^2-z^2)^{-1}$$

SUSY Adler identity

$$+ (D+\chi+w-z)(D^2-\chi^2+w^2-z^2)^{-1} L(D+w)\rangle$$

Hamiltonian hierarchies

- $N = 2n$. SUSY GD bracket is the (odd) Ham. operator $\mathcal{H} \longleftrightarrow$ conserved densities

- $N = 2n+1$ The Hamiltonian operator must be even.

- Such concept already exists in SUSY integrable community (Popowicz 09' for the Sawada-Kotera $\Leftrightarrow$ $L = D^3 + U$)

- We define even SUSY PVAs and construct such a bracket $\{\cdot\,{}_\chi\,\cdot\}_2$ on $W(\mathfrak{gl}(n+1|n), f^{pr})$ compatible with the odd SUSY PVA $\{\,{}_\chi\,\}$.

Theorem 2.

For all $a, b \in W_{2n+1}$

$$\int \phi\{\phi^{-1}(a)_\chi\, \phi^{-1}(b)\}_2 = \int \operatorname{res}\Big[L \frac{\delta a}{\delta L}\frac{\delta b}{\delta L} + (-1)^{a+1}\frac{\delta a}{\delta L} L \frac{\delta b}{\delta L} \Big]$$

Remarks

- This defines a Lie superalgebra bracket on functionals $W_{2n+1}/D\,W_{2n+1}$. We call it the even linear SUSY GD bracket.

- The hierarchy $(\partial_\ell)_{\ell\geq 1}$ on W_{2n+1} is Hamiltonian for this bracket.

(as opposed to the bi-Hamiltonian hierarchy on W_{2n})

Future extensions

- different $\mathfrak{g}$ and odd nilp. $f \in \mathfrak{g}$
- $N=2$ SUSY $\quad D_1^2 = D_2^2 = 0 \quad D = D_1 + D_2$
- Vertex algebra (both non SUSY and SUSY).

Thank you !