Zhu algebras of Generalized W algebras

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Main Theorem

1. Zhu_H(Vertex Algebra) : Associative algebra.

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2. (Choi, Molev, Suh) Constructed new class of vertex algebras, called the **generalized affine W-algebras**, and their Zhu algebras.

Overview

Vertex algebras

Definition

A vertex algebra consists of

1. a vector space V,
2. the vacuum vector $|0\rangle \in V$,
3. and n-th product $\cdot_{(n)}\cdot$, $n \in \mathbb{Z}$,

and they satisfy Vacuum axiom, Borcherds identity, and locality axiom.

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Remark

$$a_{(n)}b = \begin{cases} a_{(n)}b, n \geq 0 \\ a_{(-1)}b \\ a_{(-n)}b, n > 0 \end{cases}$$

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Remark

$$a_{(n)}b = \begin{cases} a_{(n)}b, n \geq 0 \\ a_{(-1)}b \\ a_{(-n)}b, n > 0 \end{cases} \qquad \left.\begin{array}{l} \text{Lambda bracket } [a_\lambda b] \in V[\lambda] \\ \text{Normally ordered product } :ab: \\ \text{Derivation } \partial \text{ and } :ab: \end{array}\right\}$$

Vertex algebras

More precisely,

1. The informations of $\cdot_{(n)}\cdot$, $n \geq 0$, are recorded in the lambda bracket

$$[a_\lambda b] := \sum_{n\geq 0} \frac{\lambda^n}{n!} a_{(n)} b \qquad a, b \in V,\ [a_\lambda b] \in V[\lambda]\ .$$

2. $:ab: := a_{(-1)}b$.
3. $a_{(-n-1)}b = \frac{1}{n!} :(\partial^n a)b:$ for $n \geq 0$.

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To summarize, the data of a vertex algebra can be written in the following ways : $(V, |0\rangle, \cdot_{(n)}\cdot)$, $n \in \mathbb{Z}$, or $(V, |0\rangle, [\cdot_\lambda \cdot], : :, \partial)$.

Example : Universal affine vertex algebra

Example

$\mathfrak{g}$: simple Lie algerba, $(\cdot \mid \cdot)$: the Killing form, $\{a_i\}_{i\in I}$: a basis of $\mathfrak{g}$.
For $k \in \mathbb{C}$, define

$$V^k(\mathfrak{g}) = \mathrm{Span}_{\mathbb{C}} \left\{ :(\partial^{n_1} a_{i_1}) : \cdots : \left(\partial^{n_{s-1}} a_{i_{s-1}}\right) (\partial^{n_s} a_{i_s}) ::: \middle| \; n_1 \geq n_2 \geq \cdots \geq n_s \geq 0 \right\} .$$

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Define $|0\rangle :=$ the element corresponding to $s = 0$, and

$$[a_\lambda b] = [a, b] + k\,(a|b)\,\lambda = a_{(0)}b + a_{(1)}b\lambda \ .$$

If $\mathfrak{g} = \mathfrak{sl}_2 = \mathrm{Span}_{\mathbb{C}}\{e, h, f\}$, some lambda brackets in $V^k(\mathfrak{sl}_2)$ are

$$[e_\lambda f] = h + k\lambda, \quad [h_\lambda e] = 2e, \quad [h_\lambda h] = 2k\lambda \ .$$

Hamiltonian operator

Definition

Let $(V, |0\rangle, \cdot_{(n)} \cdot)$ be a vertex algebra. A diagonalizable operator $H : V \to V$ is called a **Hamiltonian operator** if

$$\Delta(a_{(n)}b) = \Delta(a) + \Delta(b) - n - 1$$

for any eigenvectors a, b of H. Here, $\Delta(a)$ denotes the eigenvalue of a with respect to H.

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Example

In $V^k(\mathfrak{sl}_2)$, define

$$H(e) = 0, \quad H(h) = h, \quad H(f) = 2f \ .$$

Then

$$\Delta(e_{(0)}f) = \Delta(h) = 1 = 0 + 2 - 0 - 1 = \Delta(e) + \Delta(f) - 0 - 1 \ ,$$

$$\Delta(h_{(1)}h) = \Delta(2k) = 0 = 1 + 1 - 1 - 1 = \Delta(h) + \Delta(h) - 1 - 1 \ .$$

Zhu algebra

Let $(V, |0\rangle, \cdot_{(n)})$ be a vertex algebra and H be a Hamiltonian operator. The H-**tiwsted Zhu algebra** of V is defined as follows:

$$Zhu_H(V) := V/(\partial + H)V \ .$$

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Example

Let $\mathfrak{g} = \mathfrak{sl}_2$ and $H : V^k(\mathfrak{g}) \to V^k(\mathfrak{g}), H(e) = 0, H(h) = h, H(f) = 2f$ be as before. Then

$$\overline{\partial e} = \overline{-H(e)} = 0, \quad \overline{\partial h} = \overline{-H(h)} = -\overline{h}, \quad \overline{\partial f} = \overline{-H(f)} = -2\overline{f}$$

in $Zhu_H(V^k(\mathfrak{g}))$.

Zhu algebra

Theorem (Zhu, 1996)

Define a multiplication on $Zhu_H(V) = V/(\partial + H)V$ as follows :

$$\overline{a} \cdot \overline{b} := \overline{:ab:} + \sum_{j \in \mathbb{Z}_{\geq 0}} \frac{1}{j+1} \binom{\Delta(a) - 1}{j} \overline{H(a)_{(j)} b} \, .$$

Then $(Zhu_H(V), \cdot)$ is an associative algebra. Moreover,

$$\overline{a} \cdot \overline{b} - \overline{b} \cdot \overline{a} = \sum_{j \in \mathbb{Z}_{\geq 0}} \binom{\Delta(a) - 1}{j} \overline{a_{(j)} b} \, .$$

Example

Let $V = V^k(\mathfrak{sl}_2)$ and $H : V \to V$ be as before. In $Zhu_H(V)$, from

$$[e_\lambda f] = h + k\lambda, \quad [h_\lambda e] = 2e, \quad [h_\lambda h] = 2k\lambda$$

and

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we can compute as follows :

$$[\overline{e}, \overline{f}] = \overline{h} - k, \quad [\overline{h}, \overline{e}] = 2\overline{e}, \quad [\overline{h}, \overline{h}] = 0 \ .$$

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Define an algebra homomorphism

$$U(\mathfrak{sl}_2) \to Zhu_H(V^k(\mathfrak{sl}_2))$$
$$e \mapsto \overline{e}, \qquad h \mapsto \overline{h} - k, \qquad f \mapsto \overline{f}$$

then it is an algebra isomorphism.

VOA

Definition

1. A vector L in a vertex algebra V is called **conformal** if $L_{(0)} = \partial$, $L_{(1)}$ is diagonalizable, and
$$[L_\lambda L] = (\partial + 2\lambda)L + \frac{c}{12}\lambda^3 \ .$$
2. (V, L) is called a **vertex operator algebra (VOA)** if V is a vertex algebra, $L \in V$ is a conformal vector, and they satisfy some additional conditions.

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Remark

1. $\mathfrak{g}$: simple Lie algebra $\Rightarrow$ the Killing form is non-degenerate $\Rightarrow$ $\exists$ dual basis $\{a^i\}_{i\in I}$ of a given basis $\{a_i\}_{i\in I}$ of $\mathfrak{g}$.
2. If $k \neq -h^\vee$, there is a conformal vector $L = \sum_{i\in I} : a_i a^i :\in V^k(\mathfrak{g})$. This is known as the Sugawara construction.
3. If L is a conformal vector, $L_{(1)}$ is a Hamiltonian operator.

Finite W-algebras

Let N be a positive integer and from now on fix $\mathfrak{g} = \mathfrak{sl}_N(\mathbb{C})$.
Consider the left-justified pyramid corresponding to the partition $\mu = (\mu_1, \cdots, \mu_m)$ of N.
For example, for the partitions $\mu = (2, 3, 4) \vdash 9 = N$ is given by

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Define an element $e \in \mathfrak{g}$ corresponding to μ by

$$e := \sum_{\substack{i=1,\cdots,N-1 \\ \mathrm{row}_\mu(i)=\mathrm{row}_\mu(i+1)}} e_{i,i+1}$$

For example, if $\mu = (2, 3, 4)$, then $e = e_{12} + e_{34} + e_{45} + e_{67} + e_{78} + e_{89}$.

Finite W-algebras

Define a $\mathbb{Z}$-gradation on $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ as follows :

$$\deg(e_{ij}) = \mathrm{col}_\mu(j) - \mathrm{col}_\mu(i) \ .$$

For example, for $\mu = (2, 3, 4) \vdash 9 = N$, $e_{13}, e_{31}, e_{16} \in \mathfrak{g}(0)$, $e_{56} \in \mathfrak{g}(-2)$, $e_{65} \in \mathfrak{g}(2)$.

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Set $\mathfrak{n}_\mu := \bigoplus_{i>0} \mathfrak{g}(i)$ and

$$\mathcal{I}_\mu := U(\mathfrak{g}) \langle n + \chi(n) \mid n \in \mathfrak{n}_\mu \rangle$$

where $\chi \in \mathfrak{n}_\mu^*$ corresponding to $e \in \mathfrak{g}(1)$.

Finite W-algebras

Definition

The **finite W-algebra** is the associative algebra

$$U(\mathfrak{g},\mu) := (U(\mathfrak{g})/\mathcal{I}_\mu)^{\mathrm{ad}\ \mathfrak{n}_\mu} = \left(U(\mathfrak{g})/U(\mathfrak{g})\left\langle n+\chi(n) \mid n \in \mathfrak{n}_\mu\right\rangle\right)^{\mathrm{ad}\mathfrak{n}_\mu} .$$

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Example

Let $\mu = (1,1,\cdots,1)$. Then $e = 0$ and $\mathfrak{n}_\mu = 0$, so $U(\mathfrak{sl}_N,\mu) = U(\mathfrak{sl}_N)$.

Example

Let $N = 3$ and $\mu = (1,2)$. Then $e = e_{23}$ and $\mathfrak{n}_\mu = \mathrm{Span}_{\mathbb{C}}\{e_{13}, e_{23}\}$.
In this case the finite W-algebra is

$$U(\mathfrak{sl}_3,(1,2)) = \left(U(\mathfrak{sl}_3)/U(\mathfrak{sl}_3)\left\langle e_{13}, e_{23}+1\right\rangle\right)^{\mathrm{ad}\mathfrak{n}_\mu} .$$

Suppose $\mu \vdash N$ is given and $\mathfrak{n}_\mu$ as before. Define a vertex *super*algebra $\mathcal{F}(\mathfrak{n}_\mu)$, called the **free fermion vertex algebra**, as follows : as the odd vector superspaces,

$$\mathcal{F}(\mathfrak{n}_\mu) = \phi_{\mathfrak{n}_\mu} \oplus \phi^{\mathfrak{n}_\mu^*}$$

where $\phi_{\mathfrak{n}_\mu} = \{\phi_n \mid n \in \mathfrak{n}_\mu\}$, $\phi^{\mathfrak{n}_\mu^*} = \{\phi^m \mid m \in \mathfrak{n}_\mu^*\}$.

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where $\phi_{\mathfrak{n}_\mu} = \{\phi_n \mid n \in \mathfrak{n}_\mu\}$, $\phi^{\mathfrak{n}_\mu^*} = \{\phi^m \mid m \in \mathfrak{n}_\mu^*\}$. The vertex superalgebra $\mathcal{F}(\mathfrak{n}_\mu)$ is freely generated by the elements ϕ_n and ϕ^m as a differential algebra with the following λ-brackets :

$$[\phi_n\,{}_\lambda \phi^m] = m(n), \qquad [\phi_n\,{}_\lambda \phi_{n'}] = [\phi^m{}_\lambda \phi^{m'}] = 0 \ .$$

Affine W algebras

Define the vertex algebra

$$C^k(\mathfrak{g},\mu) := V^k(\mathfrak{g}) \otimes \mathcal{F}(\mathfrak{n}_\mu) = \bigoplus_{i\in\mathbb{Z}} C^k(\mathfrak{g},\mu)(i)$$

and its element

$$d := \sum_{i\in S_\mu} :\phi^{e_i^*} e_i: + \phi^\chi + \frac{1}{2}\sum_{i,i'\in S_\mu} :\phi^{e_i^*} :\phi^{e_{i'}^*}\phi_{[e_i,e_{i'}]}:: \in C^k(\mathfrak{g},\mu)(1)\ .$$

Here, $\{e_i\}_{i\in S_\mu}$ is a basis of $\mathfrak{n}_\mu$.

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Here, $\{e_i\}_{i\in S_\mu}$ is a basis of $\mathfrak{n}_\mu$.

Theorem

1. *$(d_{(0)})^2 : C^k(\mathfrak{g},\mu)(i) \to C^k(\mathfrak{g},\mu)(i+2)$ is zero.*
2. *If $i \neq 0$, $H^i(C^k(\mathfrak{g},\mu), d_{(0)}) = 0$.*

Affine W-algebras

Definition

The complex $(C^k(\mathfrak{g}, \mu), d_{(0)})$ is called the **BRST complex**.

$$W^k(\mathfrak{g}, \mu) := H^0(C^k(\mathfrak{g}, \mu), d_{(0)}) = \frac{\ker\ d_{(0)}}{\operatorname{Im}\ d_{(0)}}$$

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Theorem

1. *The conformal vector of $V^k(\mathfrak{g})$ by the Sugawara construction (with some shift) induces a conformal vector of $W^k(\mathfrak{g}, \mu)$.*
2. *(De Sole, Kac) Using the Hamiltonian operator above, $Zhu_H(W^k(\mathfrak{g}, \mu)) \cong U(\mathfrak{g}, \mu)$. In particular, $Zhu_H(V^k(\mathfrak{g})) \cong U(\mathfrak{g})$.*

Historical remarks

Let $\mathfrak{g}$ be a simple Lie algebra.

1. (Gan, Ginzburg, 2002) $U(\mathfrak{g}, \mu)$ is a quantization of Slodowy slice.
2. (De Sole, Kac, 2006) $Zhu_H(W^k(\mathfrak{g}, \mu)) \cong U(\mathfrak{g}, \mu)$.
3. (De Sole, Kac, 2006) The cardinality of the set of generators of $W^k(\mathfrak{g}, \mu)$ and $U(\mathfrak{g}, \mu)$ is the dimension of $\mathfrak{g}(0)$.
4. (Premet, 2007) The center of $U(\mathfrak{g}, \mu)$ is isomorphic to the center of $U(\mathfrak{g})$.
5. (Arakawa, 2011) The center of $W^k(\mathfrak{g}, \mu)$ is isomorphic to the center of $V^k(\mathfrak{g})$ and it is trivial unless $k = -h^\vee$.

Generalized W-algebras

From now on, we fix $\mathfrak{g} = \mathfrak{gl}_N$ and a nilpotent element $e \in \mathfrak{g}$. Define

$$\mathfrak{a} := \mathfrak{g}^e = \ker\,(\mathrm{ad}\ e) \subseteq \mathfrak{g}\ .$$

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Remarks

1. If $e = 0$, then $\mathfrak{a} = \mathfrak{gl}_N$.
2. $\mathfrak{a}$ is not reductive in general. Therefore the Killing form on $\mathfrak{a}$ is degenerate.
3. There is a symmetric invariant bilinear form on $\mathfrak{a}$, which recovers the Killing form when $e = 0$.

Generalized W-algebras

There is a basis of $\mathfrak{a}$ of the form

$$\left\{ E_{ij}^{(r)} := \sum_{\substack{\mathrm{row}_\lambda(a)=i,\mathrm{row}_\lambda(b)=j \\ \mathrm{col}_\lambda(b)-\mathrm{col}_\lambda(a)=r}} e_{ab} \;\middle|\; 1 \leq i,j \leq n \text{ and some conditions} \right\}$$

for some non-negative integer n and their relation is given by

$$[E_{ij}^{(r)}, E_{kl}^{(s)}] = \delta_{kj} E_{il}^{(r+s)} - \delta_{il} E_{kj}^{(r+s)} \ .$$

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$$[E_{ij}^{(r)}, E_{kl}^{(s)}] = \delta_{kj} E_{il}^{(r+s)} - \delta_{il} E_{kj}^{(r+s)} \ .$$

Let $\mu \vdash n$.
In the same way as before, from μ we can define a nilpotent element $\mathrm{e} \in \mathfrak{a}$ and a $\mathbb{Z}$-gradation

$$\mathfrak{a} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{a}(i) \ .$$

Generalized W-algebras

Theorem (Choi, Molev, Suh)

1. *The definition of finite and affine W-algebras for $\mathfrak{a}$ associated with μ is well-defined. These are called* **the generalized finite and affine W-algebras** *and let's denote those $U(\mathfrak{a}, \mu)$ and $W^k(\mathfrak{a}, \mu)$, respectively.*
2. *The cardinality of the set of generators of $W^k(\mathfrak{a}, \mu)$ and $U(\mathfrak{a}, \mu)$ is the dimension of $\mathfrak{a}(0)$.*
3. *When $e = 0$, then they are original W-algebras.*
4. *When $\mu = (1, 1, \cdots, 1)$, then $U(\mathfrak{a}, \mu) = U(\mathfrak{a})$.*
5. *When $\mu = (n)$, then the center of $U(\mathfrak{a})$ is isomorphic to $U(\mathfrak{a}, \mu)$.*
6. *When $\mu = (1, \cdots, 1, 2)$, we can find the explicit formula of the generators of $U(\mathfrak{a}, \mu)$ and $W^k(\mathfrak{a}, \mu)$.*

Difficulties in this algebras

Recall 1) If there is a conformal vector L in a vertex algebra, $L_{(1)}$ is a Hamiltonian operator.
Recall 2) From the Sugawara construction, $V^k(\mathfrak{g})$ has a conformal vector.
Recall 3) There is the conformal vector in $W^k(\mathfrak{g}, \mu)$ induced from the conformal vector in $V^k(\mathfrak{g})$.

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	$W^k(\mathfrak{g}, \mu)$	$W^k(\mathfrak{a}, \mu)$
Conformal vector	O	X
Hamiltonian operator	O	O

Why?

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Why? It is because $\mathfrak{a}$ does not admit a "non-degenerate" symmetric invariant bilinear form.

Further topics

1. (Gan, Ginzburg, 2002) $U(\mathfrak{g},\mu)$ is a quantization of Slodowy slice. $\Rightarrow$ What is the classical limit of $U(\mathfrak{a},\mu)$?
2. (De Sole, Kac, 2006) $Zhu_H(W^k(\mathfrak{g},\mu)) \cong U(\mathfrak{g},\mu)$. $\Rightarrow$ Also holds for $\mathfrak{a}$.
3. (De Sole, Kac, 2006) The cardinality of the set of generators of $W^k(\mathfrak{g},\mu)$ and $U(\mathfrak{g},\mu)$ is the dimension of $\mathfrak{g}(0)$.$\Rightarrow$ Also holds for $\mathfrak{a}$.
4. (Premet, 2007) The center of $U(\mathfrak{g},\mu)$ is isomorphic to the center of $U(\mathfrak{g})$. $\Rightarrow$ Also holds for $\mathfrak{a}$ and $\mu = (n)$
5. (Arakawa, 2011) The center of $W^k(\mathfrak{g},\mu)$ is isomorphic to the center of $V^k(\mathfrak{g})$ and it is trivial unless $k = -h^\vee$. $\Rightarrow$ Is it also true for $\mathfrak{a}$?