Infinite level Fock space, crystal bases, and tensor products of extremal weight modules of type $A_{+\infty}$

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joint work with Jae-Hoon Kwon, based on arxiv:2501.07941

Outline

1. Crystals
2. Fock spaces
3. Parabolic boson algebras
4. Fock space of infinite level
5. Crystal valuations and socle filtration of $\mathcal{F}^\infty \otimes \mathcal{M}$

Crystals

Reminders on crystal bases

Let us start with a brief review of crystal base theory.

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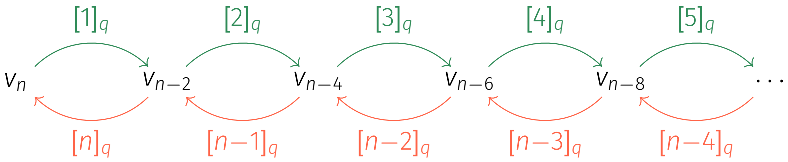
The crystal base is $q \to 0$ limit of integrable representations $V(\lambda)$ of $U_q(\mathfrak{g})$, where the limit is understood as taking a suitable *lattice* inside of $V(\lambda)$ and taking a quotient by $q = 0$.

Reminders on crystal bases

Recall that we have a following nice basis of $V(n)$ for $n \in \mathbb{Z}_{\geq 0}$.



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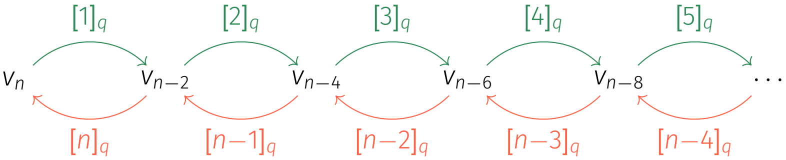

We want this structure to survive in $q \to 0$ limit. Define: $\widetilde{e}(v_i) = v_{i+2}, \widetilde{f}(v_i) = v_{i-2}$. This uniquely determines an operator for all integrable $U_q(\mathfrak{sl}_2)$-modules, called (lower) crystal operators.

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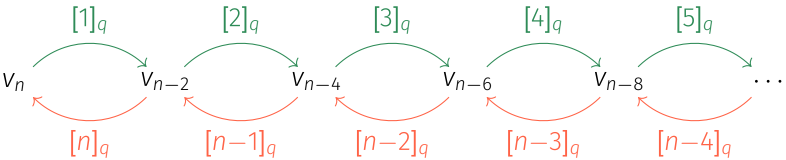

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For arbitrary $\mathfrak{g}$ with an underlying index set I, $\widetilde{e}_i, \widetilde{f}_i$s ($i \in I$) are defined via the embedding $U_{q_i}(\mathfrak{sl}_2) \hookrightarrow U_q(\mathfrak{g})$.

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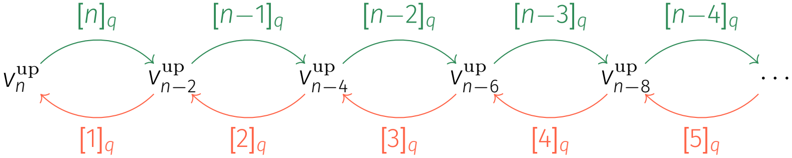

This defines (upper) crystal operators $\widetilde{e}^{\mathrm{up}}, \widetilde{f}^{\mathrm{up}}$.

Reminders on crystal bases

The construction of crystal operators are quite simple, but it requires surprisingly involved arguments to prove that there exists a lattice of $V(\lambda)$ that is stable under these operators, and that there exists a abstract crystal at $q = 0$.

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Let $A_0 = \{ f \in \mathbb{Q}(q) \,|\, f \text{ is regular at } q = 0 \}$. Let $\lambda \in P^+$ be a dominant weight.

Theorem (Kashiwara '90 for type ABCD, '91 for general case)

There exists a A_0-lattice of $V(\lambda)$, denoted $\mathcal{L}(\lambda)$, and a basis $\mathcal{B}(\lambda)$ of $\mathcal{L}(\lambda)/q\mathcal{L}(\lambda)$, such that

$$\widetilde{e}_i\mathcal{L}(\lambda) \subset \mathcal{L}(\lambda), \quad \widetilde{f}_i\mathcal{L}(\lambda) \subset \mathcal{L}(\lambda),$$

$$\widetilde{e}_i\mathcal{B}(\lambda) \subset \mathcal{B}(\lambda) \sqcup \{0\}, \quad \widetilde{f}_i\mathcal{B}(\lambda) \subset \mathcal{B}(\lambda) \sqcup \{0\},$$

$$u = \widetilde{e}_i v \iff \widetilde{f}_i u = v \quad (u, v \in \mathcal{B}(\lambda))$$

which are compatible with weight space decompositions.

More reminders on crystal bases

For an arbitrary integrable $U_q(\mathfrak{g})$-module V whose weights are finitely dominated, it decomposes into direct sum of $V(\lambda)$'s, so there exists a crystal base $(\mathcal{L}, \mathcal{B})$ for such a module.

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Theorem (Kashiwara '90, cont'd)

Crystal bases of V are unique up to isomorphism. That is, for any two crystal bases $(\mathcal{L}_i, \mathcal{B}_i)$ $(i = 1, 2)$ of V, there exists an $U_q(\mathfrak{g})$-linear automorphism $\phi : V \to V$ such that $\phi(\mathcal{L}_1) = \mathcal{L}_2$, $\overline{\phi}(\mathcal{B}_1) = \mathcal{B}_2$.

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Suppose that $V = V_1 \oplus V_2$, and that V_1 and V_2 does not share an isotypic component. Then the above, in particular, implies that $\mathcal{L}(V) = \mathcal{L}(V_1) \oplus \mathcal{L}(V_2)$.

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There are integrable modules whose set of weights are not finitely dominated, and in general, it is not known if a crystal base exists in such cases. Note that nontrivial examples arise only for $\mathfrak{g}$ of affine or indefinite type.

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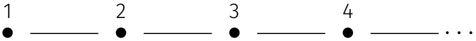
In general,

- Crystal bases of V may not be unique.
- A proper inclusion $\phi : V_1 \to V_2$ may induce an isomorphism of crystals $\overline{\phi} : \mathcal{B}_1 \to \mathcal{B}_2$.

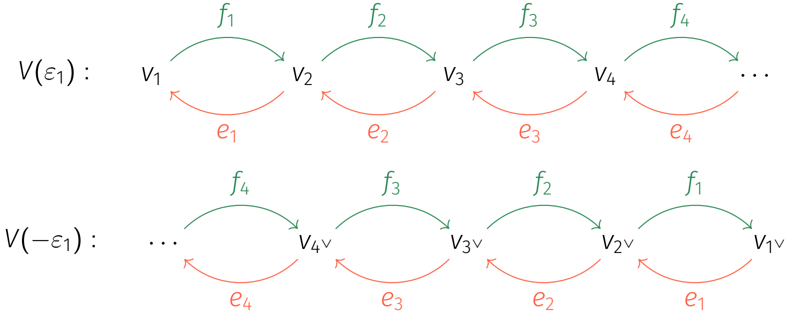
An example of non-unique crystal bases in type A_+

Consider $U_q(\mathfrak{gl}_{>0})$ associated to a Dynkin diagram A_+:



Denote its weight lattice by $P = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} \mathbb{Z}\varepsilon_i$.

The standard representation $V(\varepsilon_1)$ and its dual $V(-\varepsilon_1)$ can be described as follows.



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$\mathcal{L}(\varepsilon_1)$ and $\mathcal{L}(-\varepsilon_1)$ are free A_0-submodules generated by $\{v_i\}$ and $\{v_{i^\vee}\}$.

By tensor product rule, $V(\varepsilon_1) \otimes V(-\varepsilon_1)$ has a crystal base

$$\mathcal{L}(\varepsilon_1) \otimes \mathcal{L}(-\varepsilon_1) = \bigoplus_{i,j \in \mathbb{Z}_{\geq 0}} A_0 v_i \otimes v_{j^\vee},$$

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Note that weights of $V(\varepsilon_1) \otimes V(-\varepsilon_1)$ are not finitely dominated, and its weight 0 component has infinite dimension.

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On the other hand, consider elements $D_n \in V(\varepsilon_1) \otimes V(-\varepsilon_1)$ defined by

$$D_1 = q^{-1} v_1 \otimes v_{1^\vee},$$
$$D_2 = q^{-2} v_1 \otimes v_{1^\vee} - q^{-1} v_2 \otimes v_{2^\vee},$$
$$D_3 = q^{-3} v_1 \otimes v_{1^\vee} - q^{-2} v_2 \otimes v_{2^\vee} + q^{-1} v_3 \otimes v_{3^\vee},$$
$$D_4 = q^{-4} v_1 \otimes v_{1^\vee} - q^{-3} v_2 \otimes v_{2^\vee} + q^{-2} v_3 \otimes v_{3^\vee} - q^{-1} v_4 \otimes v_{4^\vee},$$
$$\cdots$$
$$D_n = \sum_{k=1}^{n} (-1)^{k-1} q^{-n-1+k} v_k \otimes v_{k^\vee}.$$

An example of non-unique crystal bases in type A_+

Then one can check that

$$\widetilde{e}_k D_n = \begin{cases} -\frac{(-1)^{k-1}}{1+q^2} v_n \otimes v_{(n+1)^\vee} & \text{if } k = n, \\ 0 & k \neq n, \end{cases}$$

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In particular, $\widetilde{e}_k D_n, \widetilde{f}_k D_n \in \mathcal{L}(\varepsilon_1) \otimes \mathcal{L}(-\varepsilon_1)$. Thus,

$$\mathcal{L}_{\mathbb{N}} := \mathcal{L}(\varepsilon_1) \otimes \mathcal{L}(-\varepsilon_1) + \sum_{n=1}^{\infty} A_0 D_n$$

is a crystal lattice of $V(\varepsilon_1) \otimes V(-\varepsilon_1)$.

An example of non-unique crystal bases in type A_+

Consider the image of $\mathcal{B}(\varepsilon_1) \otimes \mathcal{B}(-\varepsilon_1)$ along the inclusion

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Note that $v_1 \otimes v_{1^\vee} = qD_1$, so $1 \otimes 1^\vee \in \mathcal{B}(\varepsilon_1) \otimes \mathcal{B}(-\varepsilon_1)$ is mapped to 0. It turns out that $\mathcal{B}(\varepsilon_1) \otimes \mathcal{B}(-\varepsilon_1) = \mathcal{B}(0) \sqcup \mathcal{B}(\varepsilon_1 - \varepsilon_2)$, and the inclusion kills the connected component $\mathcal{B}(0)$.

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Thus, we got a crystal base $(\mathcal{L}_{\mathbb{N}}, \mathcal{B}(\varepsilon_1 - \varepsilon_2))$. Note that we got a *larger* A_0-lattice but a *smaller* $\mathbb{Q}$-basis.

Also, there exists an embedding $i : V(\varepsilon_1 - \varepsilon_2) \hookrightarrow V(\varepsilon_1) \otimes V(-\varepsilon_1)$, such that $i^{-1}(\mathcal{L}_{\mathbb{N}}) = \mathcal{L}(\varepsilon_1 - \varepsilon_2)$, and $\bar{i}$ induces an isomorphism of crystals.

An example of a proper inclusion inducing an isomorphism of crystal

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We use the notations of [Kas02]. Let $\mathfrak{g}$ be an affine Kac-Moody algebra with Dynkin diagram I, and let $\lambda = \sum_{i \in I_0^\vee} m_i \varpi_i$ be a dominant integral weight. Then there exists a canonical map

$$\Phi_\lambda : V(\lambda) \to \bigotimes_{i \in I_0^\vee} V(m_i \varpi_i).$$

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Theorem

1. Φ_λ *is injective.*
2. $\mathcal{L}(\lambda) = \Phi_\lambda^{-1}\left(\bigotimes_{i \in I_0^\vee} \mathcal{L}(m_i \varpi_i)\right)$.
3. $\overline{\Phi}_\lambda$ *induces an isomorphism of crystals* $\mathcal{B}(\lambda) \to \bigotimes_{i \in I_0^\vee} \mathcal{B}(m_i \varpi_i)$.

Maximal crystal lattices

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We can almost always pick a maximal one among them, with respect to the partial order by inclusion. However, we have to allow crystal lattices that are not free as an A_0-module.

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Let $\mathbb{v} : \mathbb{Q}(q) \to \mathbb{Z} \cup \{\infty\}$ be the valuation counting the order of q at 0.

A function $\mathbb{v} : V \to \mathbb{Z} \cup \{\infty\}$ on a $\mathbb{Q}(q)$-vector space V is called a valuation if

- $\mathbb{v}(v) = \infty \iff v = 0$,
- $\mathbb{v}(cv) = \mathbb{v}(c) + \mathbb{v}(v)$ for $c \in \mathbb{Q}(q)$, $v \in V$,
- $\mathbb{v}(v + w) \geq \min\{\mathbb{v}(v), \mathbb{v}(w)\}$ for $v, w \in V$.

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For an integrable $U_q(\mathfrak{g})$-module V, a valuation $\mathbb{v} : V \to \mathbb{Z} \cup \{\infty\}$ is called a crystal valuation if

- $\mathbb{v}(v) = \min(\mathbb{v}(v_\mu) \mid \mu \in P)$ for $v = \sum v_\mu$ with $v_\mu \in V_\mu$,
- $\mathbb{v}(\widetilde{e}_i v) \geq \mathbb{v}(v)$ and $\mathbb{v}(\widetilde{f}_i v) \geq \mathbb{v}(v)$ for all $v \in V$ and $i \in I$.

Maximal crystal valuations

Proposition

Suppose that V is of finite length, and $W = \operatorname{soc} V$. If $\mathbb{v}$ is a crystal valuation on V, there exists a maximal crystal valuation on V that restricts to $\mathbb{v}|_W$ on W.

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Suppose that we have $\{\mathbb{v}_s\}_{s \in S}$ for a totally ordered set S, satisfying $\mathbb{v}_s \geq \mathbb{v}_t$ whenever $s \geq t$. We claim that $\widetilde{\mathbb{v}}$ defined by

$$\widetilde{\mathbb{v}}(v) = \max \{ \mathbb{v}_s(v) \mid s \in S \}$$

is a crystal valuation.

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Only the first condition warrants proof. For any nonzero $v \in V$, there exists a nonzero $w \in U_q(\mathfrak{g})v \cap W$, then

$$w = \sum_{\boldsymbol{i}} c_{\boldsymbol{i}} \widetilde{x}_{i_1} \cdots \widetilde{x}_{i_k} v, \quad (x = e, f).$$

$\mathbb{v}(w) = \mathbb{v}_s(w) \geq \min\{\, \mathbb{v}(c_{\boldsymbol{i}}) \mid \boldsymbol{i} \,\} + \mathbb{v}_s(v)$, so $\mathbb{v}_s(v)$ is bounded above.

Saturated crystal valuations

Definition

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$\mathbb{v}$ is saturated iff $(L_\mathbb{v} \cap W)/q(L_\mathbb{v} \cap W) \to L_\mathbb{v}/qL_\mathbb{v}$ is a bijection. In particular, if $(\mathcal{L}, \mathcal{B})$ is a crystal base of V, then $\mathbb{v}_\mathcal{L}$ is saturated iff $(\mathcal{L} \cap W, \mathcal{B})$ is a crystal base of W.

Here, $L_\mathbb{v} = \{ v \in V \mid \mathbb{v}(v) \geq 0 \}$, and $\mathbb{v}_L(v) = \max \{ n \mid q^{-n} v \in L \}$.

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We simply call $\mathbb{v}$ is saturated if $\mathbb{v}$ is saturated with respect to $\operatorname{soc} V$.

Saturated crystal valuations of type A_+

Let V_{hi} and V_{lo} be integrable $U_q(\mathfrak{gl}_{>0})$-modules of highest and lowest weights, respectively.

Theorem (Kwon-L.)

There exists a saturated crystal valuation on

$$V_{hi} \otimes V_{lo} / \mathrm{soc}^d(V_{hi} \otimes V_{lo}).$$

The same result holds for $V_{lo} \otimes V_{hi}$.

Fock spaces

Fock spaces

Recall that Fock space $\mathcal{F}$ is a $\mathbb{Q}(q)$-vector space with a basis consisting of configuration of black and white dots indexed by $\mathbb{Z}$, which stabilizes to white at ∞ and to black at $-\infty$.

$$|0\rangle : \quad \cdots \quad \overset{-3}{\bullet} \quad \overset{-2}{\bullet} \quad \overset{-1}{\bullet} \quad \overset{0}{\bullet} \quad \overset{1}{\circ} \quad \overset{2}{\circ} \quad \overset{3}{\circ} \quad \cdots$$

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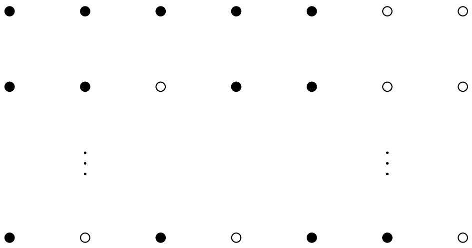
$$|0\rangle : \quad \cdots \quad \begin{matrix} -3 & -2 & -1 & 0 & 1 & 2 & 3 \\ \bullet & \bullet & \bullet & \bullet & \circ & \circ & \circ \end{matrix} \quad \cdots$$

It carries various actions of quantum groups, including our $U_q(\mathfrak{gl}_\infty)$. For example, f_i moves a black dot at i'th position to $i+1$'th position if possible, and e_i does the opposite.

This set of vectors generate a crystal lattice, and serves as a model for its crystal structure.

Commuting action on Fock space $\mathcal{F}^n$

Elements of $\mathcal{F}^n$ are depicted as below.



It carries an action of $U_p(\mathfrak{gl}_n)$ that commutes with the action of $U_q(\mathfrak{gl}_\infty)$, where the quantization parameter p is set to $-q^{-1}$. The action is given in the same way but in vertical direction.

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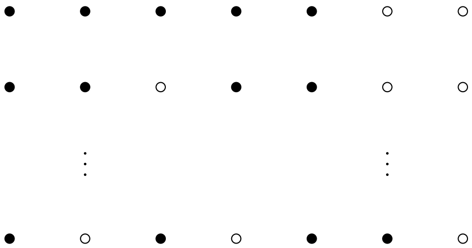

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Such an action first appeared in [Ugl00] in terms of quantum affine algebras.

Commuting action on Fock space $\mathcal{F}^n$

$\mathcal{F}^n$ is defined as certain directed limit of "double wedge spaces". Consider a standard representation $V(\varepsilon_m)$ of $U_q(\mathfrak{gl}_{\geq m})$ and $V(\dot{\varepsilon}_1)$ of $U_p(\mathfrak{gl}_n)$. Define:

$$\mathcal{A}^k(V(\varepsilon_m), V(\dot{\varepsilon}_1)) := (V(\varepsilon_m) \otimes V(\dot{\varepsilon}_1))^{\otimes k} \Big/ \sum_{i=1}^{k-1} \mathrm{im}(R_{i,i+1} - \dot{R}_{i,i+1}),$$

and $\bigwedge_{[m,\infty),n} := \bigoplus_k \mathcal{A}^k(V(\varepsilon_m), V(\dot{\varepsilon}_1))$. Here R and $\dot{R}$ are universal R-matrices for $V(\varepsilon_m)$ and $V(\dot{\varepsilon}_1)$, respectively.

Commuting action on Fock space $\mathcal{F}^n$

$\mathcal{F}^n$ is defined as certain directed limit of "double wedge spaces". Consider a standard representation $V(\varepsilon_m)$ of $U_q(\mathfrak{gl}_{\geq m})$ and $V(\dot{\varepsilon}_1)$ of $U_p(\mathfrak{gl}_n)$. Define:

$$\mathcal{A}^k(V(\varepsilon_m), V(\dot{\varepsilon}_1)) := (V(\varepsilon_m) \otimes V(\dot{\varepsilon}_1))^{\otimes k} \Big/ \sum_{i=1}^{k-1} \mathrm{im}(R_{i,i+1} - \dot{R}_{i,i+1}),$$

and $\bigwedge_{[m,\infty),n} := \bigoplus_k \mathcal{A}^k(V(\varepsilon_m), V(\dot{\varepsilon}_1))$. Here R and $\dot{R}$ are universal R-matrices for $V(\varepsilon_m)$ and $V(\dot{\varepsilon}_1)$, respectively.

Then $\mathcal{F}^n$ is defined to be a directed limit of

$$\begin{array}{ccc} \bigwedge_{[m,\infty),n} & \longrightarrow & \bigwedge_{[m-1,\infty),n} \\ w & \longmapsto & w \wedge w_{\{m\}\times[1,n]} \end{array}$$

as $m \to -\infty$. Here $w_{\{m\}\times[1,n]}$ denotes an element of $\bigwedge_{[m-1,\infty),n}$ that has "black dots in coordinates $\{m\} \times [1,n]$".

Crystal base of $\mathcal{F}^n$

There are some caveats on defining crystal operators on $\mathcal{F}^n$. We need crystal operators on $\mathcal{F}^n$ with respect to $U_p(\mathfrak{gl}_n)$ as well as $U_q(\mathfrak{gl}_\infty)$ that commutes with each other.

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- For $U_p(\mathfrak{gl}_n)$, we use a pullback of $\dot{\widetilde{e}}^{\mathrm{up}}, \dot{\widetilde{f}}^{\mathrm{up}}$ under an isomorphism of $\mathbb{Q}$-algebras $\psi : U_q(\mathfrak{gl}_n) \to U_p(\mathfrak{gl}_n)$, sending $e_i, f_i \mapsto e_i, f_i$, $q^h \mapsto p^{-h}$, and $q \mapsto p^{-1}$.

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Because p is not mapped to $-q^{-1}$, this does not extends to an algebra morphism $U_q(\mathfrak{gl}_\infty) \otimes U_q(\mathfrak{gl}_n) \to U_q(\mathfrak{gl}_\infty) \otimes U_p(\mathfrak{gl}_n)$, but nevertheless, these give commuting crystal operators on $\mathcal{F}^n$, and $\mathcal{L}(\mathcal{F}^n)$ are stable under these operators.

Semisimple decomposition of $\mathcal{F}^n$

Having a crystal base $(\mathcal{L}(\mathcal{F}^n), \mathcal{B}(\mathcal{F}^n))$, a standard argument using combinatorics of $\mathcal{B}(\mathcal{F}^n)$ yields a decomposition:

$$\mathcal{F}^n \cong \bigoplus_{\lambda \in \mathbb{Z}_+^n} V(\Lambda_\lambda) \otimes V(\dot{\varepsilon}_\lambda),$$

$$\mathcal{B}(\mathcal{F}^n) \cong \bigsqcup_{\lambda \in \mathbb{Z}_+^n} \mathcal{B}(\Lambda_\lambda) \otimes \mathcal{B}(\dot{\varepsilon}_\lambda).$$

This can be seen as a quantum analogue of the Howe duality, or level-rank duality.

Here, $\mathbb{Z}_+^n$ is the set of integer partitions of length n, and

$$\Lambda_\lambda = \sum_{i=1}^{\ell(\lambda)} \Lambda_{\lambda_i}, \varepsilon_\lambda = \sum_{i=1}^{\ell(\lambda)} \varepsilon_{\lambda_i}$$

For $\mu, \nu \in \mathcal{P}$, Let λ_n be an integer partition of length n obtained by joining μ and $-\nu$, and $\lambda = \lambda_{\ell(\mu)+\ell(\nu)}$. We also write:
$\Lambda_{\mu,\nu} = \Lambda_\lambda, \varepsilon^n_{\mu,\nu} = \varepsilon_{\lambda_n}, \varepsilon_{\mu,\nu} = \varepsilon_\lambda$.

Embedding extremal weight modules into Fock spaces

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It was observed in [Kwo09] and [Kwo11] that the abstract crystal $\mathcal{B}(\mathcal{F}^n)$ admits a limit, which was used to compute decomposition numbers of tensor products of extremal weight crystals for type $A_{+\infty}$ and A_∞.

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It was also observed in loc. cit. that $\mathcal{B}(\varepsilon_{\emptyset,\nu}) \otimes \mathcal{B}(\varepsilon_{\mu,\emptyset}) \cong \mathcal{B}(\varepsilon_{\mu,\nu})$, but $\mathcal{B}(\varepsilon_{\mu,\emptyset}) \otimes \mathcal{B}(\varepsilon_{\emptyset,\nu})$ only contains $\mathcal{B}(\varepsilon_{\mu,\nu})$ as a connected component. Note that this implies that $\mathcal{L}(\varepsilon_{\emptyset,\nu}) \otimes \mathcal{L}(\varepsilon_{\mu,\emptyset})$ is a saturated crystal lattice with respect to $V(\varepsilon_{\mu,\nu})$.

Parabolic boson algebras

Boson algebras and $\mathcal{B}(\infty)$

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It is generated by e'_i, f_i ($i \in I$) subject to the following relations:

$$e'_i f_j = q_i^{\langle h_i, \alpha_j \rangle} f_j e'_i + \delta_{ij}, \quad \mathrm{Serre}_{ij}(e'_i, e'_j) = \mathrm{Serre}_{ij}(f_i, f_j) = 0$$

where

$$\mathrm{Serre}_{ij}(x, y) = \sum_{k+l=-\langle h_i, \alpha_j \rangle - 1} (-1)^k x^{(k)} y x^{(l)}$$

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$$e_i' f_j = q_i^{\langle h_i, \alpha_j \rangle} f_j e_i' + \delta_{ij}, \quad \mathrm{Serre}_{ij}(e_i', e_j') = \mathrm{Serre}_{ij}(f_i, f_j) = 0$$

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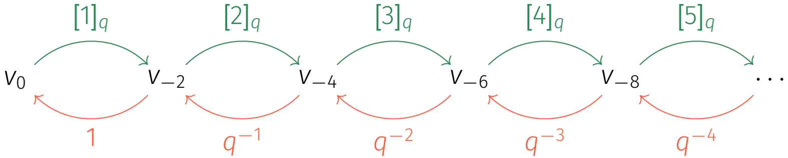
There exists an action of $B_q(\mathfrak{g})$ on $U_q^-(\mathfrak{g})$. In fact, any “finitely dominated” representation of $B_q(\mathfrak{g})$ are just direct sums of $U_q^-(\mathfrak{g})$’s.

Boson algebras: $\mathfrak{sl}_2$ case

To define crystal operators for $B_q(\mathfrak{sl}_2)$-modules, we use the following basis:



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We define $U_q(\mathfrak{g}, \mathfrak{p})$ to be an associative $\mathbb{Q}(q)$-algebra generated by e'_i, e_j, f_l, q^h for $i \in J^c, j \in J, l \in I$, and $h \in P_J^\vee$ with relations:

$$e_j f_l - f_l e_j = \delta_{jl} \frac{t_j - t_j^{-1}}{q_j - q_j^{-1}}, \quad e'_i f_l = q^{-\langle h_i, \alpha_l \rangle} f_l e'_i + \delta_{il},$$

$$\mathrm{Serre}_{i_1,i_2}(e'_{i_1}, e'_{i_2}) = \mathrm{Serre}_{l_1,l_2}(f_{l_1}, f_{l_2}) = \mathrm{Serre}_{j_1,j_2}(e_{j_1}, e_{j_2}) = 0,$$

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$$\mathrm{Serre}^\pm_{i,j}(x, y) = \sum_{k+l=-\langle h_i, \alpha_j \rangle - 1} (-1)^k q_i^{\pm k \langle h_i, \alpha_j \rangle} x^{(k)} y x^{(l)}$$

$U_q^-(\mathfrak{g})$-comodule structure

There exists an algebra homomorphism:

$$\begin{aligned}\Delta : U_q(\mathfrak{g},\mathfrak{p}) &\longrightarrow U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g},\mathfrak{p})\\ q^h &\longmapsto q^h \otimes q^h,\\ e_i' &\longmapsto -(q_i - q_i^{-1})t_i e_i \otimes 1 + t_i \otimes e_i',\\ e_j &\longmapsto e_j \otimes t_j^{-1} + 1 \otimes e_j,\\ f_l &\longmapsto f_l \otimes 1 + t_l \otimes f_l.\end{aligned}$$

Thus, given a $U_q(\mathfrak{g},\mathfrak{p})$-module V and $U_q(\mathfrak{g})$-module W, $V \otimes W$ has a natural structure of a $U_q(\mathfrak{g},\mathfrak{p})$-module.

Integrable modules of $U_q(\mathfrak{g}, \mathfrak{p})$

Let $\mathcal{O}$ be the category of $U_q(\mathfrak{g}, \mathfrak{p})$-modules V such that

1. V has a weight space decomposition with respect to $U_q^0(\mathfrak{g}, \mathfrak{p})$,
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For $\lambda \in P_J^+$, the parabolic Verma module $V_J(\lambda) = U_q^-(\mathfrak{g}) \otimes_{U_q^-(\mathfrak{l})} V_{\mathfrak{l}}(\lambda)$ carries an action of $U_q(\mathfrak{g}, \mathfrak{p})$.

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Theorem (Complete reducibility, Kwon-L.)

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We use an analogue of quantum Casimir operator Ω, which lives in a certain completion of $U_q(\mathfrak{g},\mathfrak{p})$.

Crystal bases of integrable $U_q(\mathfrak{g}, \mathfrak{p})$-modules

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- Crystal bases of an object of $\mathcal{O}^{\mathrm{int}}$ are unique up to isomorphism.

Fock space of infinite level

A directed system

Consider a parabolic subalgebra of $\mathfrak{sl}_\infty$ corresponding to a subset $J = \mathbb{Z} \setminus \{0\}$. We denote this parabolic boson algebra by $U_q(\mathfrak{sl}_{\infty,0})$, and denote V_J, $\mathcal{L}_J$, $\mathcal{B}_J$ by V_0, $\mathcal{L}_0$, $\mathcal{B}_0$, respectively.

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$\mathcal{F} \otimes \mathcal{M}$ has an action of $U_q(\mathfrak{sl}_{\infty,0})$, and $|0\rangle \otimes 1$ is a highest weight vector of weight 0 (under the quotient $P \to P_J$). Thus, $\mathcal{F} \otimes \mathcal{M}$ has a direct summand $\mathcal{M}$, and there exists an inclusion

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By applying $\mathcal{F}^n \otimes -$ to the left, we get a directed system of $U_q(\mathfrak{sl}_{\infty,0})$-modules $\phi_n : \mathcal{F}^n \otimes \mathcal{M} \to \mathcal{F}^{n+1} \otimes \mathcal{M}$.

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$\mathcal{F}^n \otimes \mathcal{M}$ has a crystal base $\mathcal{L}(\mathcal{F}^n \otimes \mathcal{M}) = \mathcal{L}(\mathcal{F}^n) \otimes \mathcal{L}(\mathcal{M})$ and $\mathcal{B}(\mathcal{F}^n \otimes \mathcal{M}) = \mathcal{B}(\mathcal{F}^n) \otimes \mathcal{B}(\mathcal{M})$. The directed system is compatible with the them, so we get a limit $(\mathcal{L}(\mathcal{F}^\infty \otimes \mathcal{M}), \mathcal{B}(\mathcal{F}^\infty \otimes \mathcal{M}))$.

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This crystal is isomorphic to a limit of $\mathcal{B}(\mathcal{F}^n)$ considered in [Kwo09], and in particular,

$$\mathcal{B}(\mathcal{F}^\infty \otimes \mathcal{M}) \cong \bigsqcup_{\mu,\nu\in\mathcal{P}} \mathcal{B}_0(\Lambda_{\mu,\nu}) \otimes \mathcal{B}(\dot{\varepsilon}_{\mu,\nu}).$$

Isotypic decomposition of $\mathcal{F}^\infty \otimes \mathcal{M}$

We exploit the semisimplicity of $U_q(\mathfrak{sl}_{\infty,0})$ to transfer structure of $\mathcal{F}^\infty \otimes \mathcal{M}$ to $U_p(\mathfrak{gl}_{>0})$-modules. The isotypic decomposition of $\mathcal{F}^\infty \otimes \mathcal{M}$ is a starting point.

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Theorem (Kwon-L.)

$$\mathcal{F}^\infty \otimes \mathcal{M} \cong \bigoplus_{\mu,\nu \in \mathcal{P}} V_0(\Lambda_{\mu,\nu}) \otimes \left(V(\dot{\varepsilon}_{\emptyset,\nu}) \otimes V(\dot{\varepsilon}_{\mu,\emptyset}) \right).$$

Here, all $\otimes$ are lower comultiplications.

Note that the multiplicity space is 'larger' than $V(\dot{\varepsilon}_{\mu,\nu})$.

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$v_{\Lambda_{\mu,\nu}} \otimes v_{\dot{\varepsilon}_{\mu,\nu}}$ corresponds to a unique highest weight element of $\mathcal{B}(\mathcal{F}^{\ell(\mu)+\ell(\nu)})$ with weight $(\Lambda_{\mu,\nu}, \dot{\varepsilon}_{\mu,\nu})$.

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We let $(\mathcal{F}^\infty \otimes \mathcal{M})_0$ be the submodule generated by lifts of these crystal basis elements. Then

$$(\mathcal{F}^\infty \otimes \mathcal{M})_0 \cong \bigoplus_{\mu,\nu \in \mathcal{P}} V_0(\Lambda_{\mu,\nu}) \otimes V(\dot{\varepsilon}_{\mu,\nu}),$$

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Isotypic decomposition of $\mathcal{F}^\infty \otimes \mathcal{M}$

Indeed, it is easy to describe a submodule $(\mathcal{F}^\infty \otimes \mathcal{M})_0$ which realizes multiplicity spaces $V(\dot{\varepsilon}_{\mu,\nu})$.In the isomorphism

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Thus, a proper inclusion $(\mathcal{F}^\infty \otimes \mathcal{M})_0 \hookrightarrow (\mathcal{F}^\infty \otimes \mathcal{M})$ induces an isomorphism of crystals.

Socle of $\mathcal{F}^\infty \otimes \mathcal{M}$

This fits into our earlier framework: $\mathcal{L}(\mathcal{F}^\infty \otimes \mathcal{M})$ is *saturated* with respect to $(\mathcal{F}^\infty \otimes \mathcal{M})_0$.

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Recall that it is natural to consider saturated crystal lattices with respect to a socle, but in this case, we do not (yet) know if $(\mathcal{F}^\infty \otimes \mathcal{M})_0$ is a socle of $\mathcal{F}^\infty \otimes \mathcal{M}$.

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It turned out that, the existence of a saturated crystal lattice is a pretty strong constraint, and we can prove that $(\mathcal{F}^\infty \otimes \mathcal{M})_0$ is a socle of $\mathcal{F}^\infty \otimes \mathcal{M}$ from this fact. We are reversing the order of arguments! The compatibility of crystal bases with isotypic component decomposition (and its version for $U_q(\mathfrak{sl}_{\infty,0}) \otimes U_p(\mathfrak{gl}_n)$) is crucial here.

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By passing to multiplicity spaces, this implies

$$\mathrm{soc}(V(\varepsilon_{\mu,\emptyset}) \otimes V(\varepsilon_{\emptyset,\nu})) = V(\varepsilon_{\mu,\nu}).$$

Crystal valuations and socle filtration of $\mathcal{F}^{\infty} \otimes \mathcal{M}$

Socle filtration of $\mathcal{F}^\infty \otimes \mathcal{M}$

We pushed this idea further, and constructed a filtration $(\mathcal{F}^\infty \otimes \mathcal{M})_{\geq -d}$ ($d \geq 0$) of $\mathcal{F}^\infty \otimes \mathcal{M}$.

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Theorem (Kwon-L.)

The subquotients of the filtration is given by

$$\frac{(\mathcal{F}^\infty \otimes \mathcal{M})_{\geq -d}}{(\mathcal{F}^\infty \otimes \mathcal{M})_{> -d}} \cong \bigoplus_{\substack{(\mu,\nu)\leq(\zeta,\eta) \\ |\zeta|-|\mu|=|\eta|-|\nu|=d}} V_0(\Lambda_{\zeta,\eta}) \otimes V(\dot{\varepsilon}_{\mu,\nu})^{\oplus n^{\mu,\nu}_{\zeta,\eta}}$$

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Here, we are considering a partial order on $\mathcal{P}^2$ defined by

$$(\mu,\nu) \leq (\zeta,\eta) \iff \zeta \supset \mu, \eta \supset \nu, \text{ and } |\zeta| - |\mu| = |\eta| - |\nu|.$$

Also,

$$n^{\mu,\nu}_{\zeta,\eta} = \sum_\sigma c^\mu_{\sigma,\zeta} c^\nu_{\sigma,\eta},$$

where $c^\mu_{\sigma,\zeta}$ is the Littlewood-Richardson coefficient.

Saturated crystal valuations on socle quotients of $\mathcal{F}^\infty \otimes \mathcal{M}$

Moreover, we constructed a crystal valuation $\mathbb{v}_{-d}^{\infty}$ on $\dfrac{\mathcal{F}^\infty \otimes \mathcal{M}}{(\mathcal{F}^\infty \otimes \mathcal{M})_{>-d}}$.

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$\mathbb{v}_{-d}^{\infty}$ *is saturated with respect to* $\dfrac{(\mathcal{F}^\infty \otimes \mathcal{M})_{\geq -d}}{(\mathcal{F}^\infty \otimes \mathcal{M})_{>-d}}$.

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Theorem (Kwon-L.)

$\mathbb{v}^\infty_{-d}$ is saturated with respect to $\dfrac{(\mathcal{F}^\infty \otimes \mathcal{M})_{\geq -d}}{(\mathcal{F}^\infty \otimes \mathcal{M})_{>-d}}$.

Corollary

$(\mathcal{F}^\infty \otimes \mathcal{M})_{\geq -d}$ the socle filtration of $\mathcal{F}^\infty \otimes \mathcal{M}$.

Construction of $(\mathcal{F}^\infty \otimes \mathcal{M})_{\geq -d}$

Each $\mathcal{F}^n \otimes \mathcal{M}$ are semisimple; It admits the following isotypic decomposition.

$$\mathcal{F}^n \otimes \mathcal{M} \cong \bigoplus_{(\mu,\nu),(\zeta,\eta)\in\mathcal{P}^2} V_0(\Lambda_{\zeta,\eta}) \otimes V(\dot{\varepsilon}^n_{\mu,\nu})^{h^{n,(\mu,\nu)}_{(\zeta,\eta)}},$$

where the multiplicity $h^{n,(\mu,\nu)}_{(\zeta,\eta)}$ is read off from crystal:

$$h^{n,(\mu,\nu)}_{(\zeta,\eta)} = \#\Big\{ \text{highest weight elements of } \mathcal{B}(\mathcal{F}^n \otimes \mathcal{M}) \text{ of weight } \Lambda_{\zeta,\eta}, \text{ whose } \mathcal{B}(\mathcal{F}^n)\text{-component is } M^n(\mu,\nu)\Big\}.$$

Let us denote the above isotypic component by $(\mathcal{F}^n \otimes \mathcal{M})^{(\zeta,\eta)}_{(\mu,\nu)}$.

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Lemma

1. $(\mathcal{F}^n \otimes \mathcal{M})^{(\zeta,\eta)}_{(\mu,\nu)} \neq 0$ *only if* $(\zeta,\eta) \geq (\mu,\nu)$.
2. $(\mathcal{F}^n \otimes \mathcal{M})^{(\zeta,\eta)}_{(\mu,\nu)} \subset (\mathcal{F}^{n+1} \otimes \mathcal{M})^{(\zeta,\eta)}_{\geq(\mu,\nu)}$.

Here, $(\mathcal{F}^n \otimes \mathcal{M})^{(\zeta,\eta)}_{\geq(\mu,\nu)} = \bigoplus_{(\sigma,\tau)\geq(\mu,\nu)} (\mathcal{F}^n \otimes \mathcal{M})^{(\zeta,\eta)}_{(\sigma,\tau)}$.

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Therefore, a directed system $\left\{(\mathcal{F}^n \otimes \mathcal{M})^{(\zeta,\eta)}_{\geq(\mu,\nu)}\right\}$ is well-defined, whose limit is a subset of $\mathcal{F}^\infty \otimes \mathcal{M}$, denoted by $(\mathcal{F}^\infty \otimes \mathcal{M})^{(\zeta,\eta)}_{\geq(\mu,\nu)}$.

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Finally, for $d \in \mathbb{Z}_{\geq 0}$,

$$(\mathcal{F}^n \otimes \mathcal{M})_{\geq -d} := \bigoplus_{\substack{(\zeta,\eta)\geq(\mu,\nu) \\ |\zeta|-|\mu|=|\eta|-|\nu|\leq d}} (\mathcal{F}^n \otimes \mathcal{M})^{(\zeta,\eta)}_{\geq(\mu,\nu)}.$$

Constructing crystal valuation (or reversing arguments once more)

In case of $d = 0$, we already had a crystal lattice of $\mathcal{F}^\infty \otimes \mathcal{M}$, this is the virtue of Fock spaces. Let's analyze its behavior in more detail.

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$$\overline{\phi} : \mathcal{B}(\mathcal{M}) \to \mathcal{B}(\mathcal{F}) \otimes \mathcal{B}(\mathcal{M})$$

increases the weight of the $\mathcal{B}(\mathcal{M})$, except for $1 \in \mathcal{B}(\mathcal{M})$. Thus, every element of $\mathcal{B}(\mathcal{F}^\infty \otimes \mathcal{M})$ has a representative of a form $b \otimes 1 \in \mathcal{B}(\mathcal{F}^n \otimes \mathcal{M})$ for sufficiently large n.

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Since $b \otimes 1$ is connected to a highest weight element of a form $b' \otimes 1$, it is contained in a connected component isomorphic to $\mathcal{B}(\Lambda_{\zeta,\eta}) \otimes \mathcal{B}(\dot{\varepsilon}^n_{\zeta,\eta})$.

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$$\pi^n_{-d} : \mathcal{F}^n \otimes \mathcal{M} \to (\mathcal{F}^n \otimes \mathcal{M})_{-d} := \bigoplus_{\substack{(\zeta,\eta)\geq(\mu,\nu) \\ |\zeta|-|\mu|=|\eta|-|\nu|=d}} (\mathcal{F}^n \otimes \mathcal{M})^{(\zeta,\eta)}_{(\mu,\nu)}$$

be the projection onto isotypic components.

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be the projection onto isotypic components.

Observation

For any $x \in \mathcal{L}(\mathcal{F}^\infty \otimes \mathcal{M})$,

$$\pi^n_0(x) \in \mathcal{L}(\mathcal{F}^n \otimes \mathcal{M}), \quad \pi^n_{-d}(x) \in q\mathcal{L}(\mathcal{F}^n \otimes \mathcal{M}) \quad (d > 0),$$

for all sufficiently large n.

Constructing crystal valuation (or reversing arguments once more)

Let $\mathbb{v}$ be the crystal valuation of $\mathcal{F}^\infty \otimes \mathcal{M}$ associated to $\mathcal{L}(\mathcal{F}^\infty \otimes \mathcal{M})$. Let

$$\mathbb{v}^n_{-d} := \mathbb{v} \circ \pi^n_{-d}.$$

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We conjectured that this pattern continues.

Defining $\mathbb{v}^{\infty}_{-d}$ as a limit

Theorem (Kwon-L.)

Let $x \in \mathcal{F}^{\infty} \otimes \mathcal{M}$ be given. Then the limit

$$\mathbb{v}^{\infty}_{-d}(x) := \lim_{n \to \infty} \left(\mathbb{v}^{n}_{-d}(x) - dn \right)$$

exists and lies in $\mathbb{Z} \sqcup \{\infty\}$. Moreover, $\mathbb{v}^{\infty}_{-d}(x)$ is finite if and only if $x \notin (\mathcal{F}^{\infty} \otimes \mathcal{M})_{>-d}$.

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Corollary

$\mathbb{v}^{\infty}_{-d}$ *induces a well-defined crystal valuation on* $\dfrac{\mathcal{F}^\infty \otimes \mathcal{M}}{(\mathcal{F}^\infty \otimes \mathcal{M})_{>-d}}$.

Remark on the proof of the theorem

Its proof is based on the following key lemmas controlling the asymptotic growth of $\mathbb{v}_d^n(x)$. Let $(\zeta, \eta) \in \mathcal{P}^2$ be given.

Lemma

1. *For all sufficiently large n, for all $x \in (\mathcal{F}^n \otimes \mathcal{M})_{-d}^{(\zeta,\eta)}$, we have*

$$\mathbb{v}_{-d}^{n+1}(x) = \mathbb{v}_{-d}^{n}(x) + d.$$

2. *For all sufficiently large n and N, for all $x \in (\mathcal{F}^n \otimes \mathcal{M})_{-d}^{(\zeta,\eta)}$, we have*

$$\mathbb{v}_{-d+1}^{n+N}(x) \in \mathbb{v}_{-d}^{n}(x) + N(d-1) + [r_1, r_2],$$

where $[r_1, r_2]$ is an interval independent of x, n and N, depending only on (ζ, η).

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These lemmas make heavy use of crystal base theory and the theory of parabolic boson algebras.

Parabolic q-derivations

The second statement essentially follows from a non-vanishing property of parabolic analogue of q-derivations.

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There exists a $U_q(\mathfrak{g})$-linear map $\mathcal{M}_J \to \mathcal{M}_J \otimes \mathcal{M}_J$ sending $1 \mapsto 1 \otimes 1$.

$$r_i^{\pm} : \mathcal{M}_J \longrightarrow \mathcal{M}_J \otimes_{\pm} \mathcal{M}_J \xrightarrow{1\otimes\pi_i} \mathcal{M}_J \otimes_{\pm} V_{\mathfrak{l}}(-\alpha_i),$$

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These are direct analogues of q-derivations in [Lus10], or operators e_i', e_i'''s in [Kas91].

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These are direct analogues of q-derivations in [Lus10], or operators e_i', e_i'''s in [Kas91].

Lemma

If $u \in \mathcal{M}_J$ satisfies $r_i^+(u) = 0$ for all $i \in J^c$, then u is a scalar multiple of 1. The same holds for r_i^-, ${}_i r^+$, and ${}_i r^-$.

Saturatedness of $\mathbb{v}^{\infty}_{-d}$

Theorem (Kwon-L.)

The crystal valuation $\mathbb{v}^{\infty}_{-d}$ *on* $\dfrac{\mathcal{F}^{\infty} \otimes \mathcal{M}}{(\mathcal{F}^{\infty} \otimes \mathcal{M})_{>-d}}$ *is saturated with respect to* $\dfrac{(\mathcal{F}^{\infty} \otimes \mathcal{M})_{\geq -d}}{(\mathcal{F}^{\infty} \otimes \mathcal{M})_{>-d}}$.

By passing to multiplicity spaces, we obtain the corollary that $(\mathcal{F}^{\infty} \otimes \mathcal{M})_{\geq -d}$ is the socle filtration of $\mathcal{F}^{\infty} \otimes \mathcal{M}$.

Remark on possibly non-free A_0 modules

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We needed compatibility of crystal bases with isotypic decomposition in the proof that 'an existence of saturated crystal valuation implies socle'. We have verified that this still holds in crystal valuations, without freeness, and even without the existence of a basis $\mathcal{B}$.

Passing to multiplicity spaces

Corollary

The Loewy length of a $U_q(\mathfrak{gl}_{>0})$-module $V(\varepsilon_{\mu,\emptyset}) \otimes V(\varepsilon_{\emptyset,\nu})$ is $\min(|\mu|, |\nu|) + 1$*, and its subquotients are given by*

$$\frac{\mathrm{soc}^{d+1}\left(V(\varepsilon_{\mu,\emptyset}) \otimes V(\varepsilon_{\emptyset,\nu})\right)}{\mathrm{soc}^{d}\left(V(\varepsilon_{\mu,\emptyset}) \otimes V(\varepsilon_{\emptyset,\nu})\right)} \cong \bigoplus_{\substack{(\mu,\nu)\geq(\zeta,\eta) \\ |\mu|-|\zeta|=|\nu|-|\eta|=d}} V_{\zeta,\eta}^{\oplus n_{\zeta,\eta}^{\mu,\nu}},$$

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Corollary

The Loewy length of a $U_q(\mathfrak{gl}_{>0})$*-module* $V(\varepsilon_{\mu,\emptyset}) \otimes V(\varepsilon_{\emptyset,\nu})$ *is* $\min(|\mu|, |\nu|) + 1$, *and its subquotients are given by*

$$\frac{\mathrm{soc}^{d+1}\left(V(\varepsilon_{\mu,\emptyset}) \otimes V(\varepsilon_{\emptyset,\nu})\right)}{\mathrm{soc}^{d}\left(V(\varepsilon_{\mu,\emptyset}) \otimes V(\varepsilon_{\emptyset,\nu})\right)} \cong \bigoplus_{\substack{(\mu,\nu)\geq(\zeta,\eta) \\ |\mu|-|\zeta|=|\nu|-|\eta|=d}} V_{\zeta,\eta}^{\oplus n_{\zeta,\eta}^{\mu,\nu}},$$

Corollary

The Loewy length of a $U_q(\mathfrak{gl}_{>0})$*-module* $V(\varepsilon_{\alpha,\beta}) \otimes V(\varepsilon_{\gamma,\delta})$ *is* $\min(|\alpha| + |\gamma|, |\beta| + |\delta|) + 1$, *and its subquotients are given by*

$$\frac{\mathrm{soc}^{d+1}\left(V(\varepsilon_{\alpha,\beta}) \otimes V(\varepsilon_{\gamma,\delta})\right)}{\mathrm{soc}^{d}\left(V(\varepsilon_{\alpha,\beta}) \otimes V(\varepsilon_{\gamma,\delta})\right)} \cong \bigoplus_{\substack{\phi,\psi\in\mathcal{P} \\ |\phi|=M-d,\ |\psi|=N-d}} V_{\phi,\psi}^{\oplus c_{(\alpha,\beta)(\gamma,\delta)}^{(\phi,\psi)}},$$

Passing to multiplicity spaces

Corollary

$\dfrac{V(\varepsilon_{\mu,\emptyset}) \otimes V(\varepsilon_{\emptyset,\nu})}{\mathsf{soc}^d\left(V(\varepsilon_{\mu,\emptyset}) \otimes V(\varepsilon_{\emptyset,\nu})\right)}$ *has a saturated crystal valuation.*

Passing to multiplicity spaces

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$$\frac{V(\varepsilon_{\mu,\emptyset}) \otimes V(\varepsilon_{\emptyset,\nu})}{\mathsf{soc}^d \left(V(\varepsilon_{\mu,\emptyset}) \otimes V(\varepsilon_{\emptyset,\nu})\right)}$$ *has a saturated crystal valuation.*

Putting $\mu = \nu = (1)$ and $d = 0$ to the last corollary recovers the saturated crystal valuation $\mathcal{L}_{\mathbb{N}}$ of $V(\varepsilon_1) \otimes V(-\varepsilon_1)$ constructed in the first section of this talk.

Passing to multiplicity spaces

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Putting $\mu = \nu = (1)$ and $d = 0$ to the last corollary recovers the saturated crystal valuation $\mathcal{L}_{\mathbb{N}}$ of $V(\varepsilon_1) \otimes V(-\varepsilon_1)$ constructed in the first section of this talk.

We do not (yet) know whether saturated crystal valuations exist in $\dfrac{V(\varepsilon_{\alpha,\beta}) \otimes V(\varepsilon_{\gamma,\delta})}{\mathrm{soc}^d\left(V(\varepsilon_{\alpha,\beta}) \otimes V(\varepsilon_{\gamma,\delta})\right)}$ or not, but low rank calculations suggests that they do.

Final remarks

- Beyond low rank cases, we do not have an effective algorithm to determine whether a given element belongs to the A_0-submodule or not.

Final remarks

- Beyond low rank cases, we do not have an effective algorithm to determine whether a given element belongs to the A_0-submodule or not.
- We expect that there are abundance of saturated crystal valuations. It would be interesting to study them for other infinite affine or affine types, which will provide us a better understanding of their internal structures.

Thank you for your attention!

Jonathan Beck and Hiraku Nakajima.
Crystal bases and two-sided cells of quantum affine algebras.
Duke mathematical journal, 123(2):335–402, 2004.

M. Kashiwara.
On crystal bases of the q-analogue of universal enveloping algebras.
Duke Mathematical Journal, 63(2):465 – 516, 1991.

Masaki Kashiwara.
On level-zero representations of quantized affine algebras.
Duke mathematical journal, 112(1):117–175, 2002.

Jae-Hoon Kwon.
Differential operators and crystals of extremal weight modules.
Advances in mathematics (New York. 1965), 222(4):1339–1369, 2009.

Jae-Hoon Kwon.
Crystal duality and littlewood–richardson rule of extremal weight crystals.
Journal of algebra, 336(1):99–138, 2011.

George Lusztig.
Introduction to quantum groups.
Modern Birkhäuser classics. Birkhäuser, Boston, 2010.

D Uglov.
Canonical bases of higher-level q-deformed fock spaces and kazhdan-lusztig polynomials.
In *PHYSICAL COMBINATORICS*, volume 191, pages 249–299, CAMBRIDGE, 2000. Birkhauser Boston.