Introduction to Galois Cohomology

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Goals of this talk

Introduce Galois cohomology and provide two applications for Prof. Kim's talk.

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Theorem A (Kummer Theory)

Let K be a number field and suppose that $\mu_n \subset K$. Then we have an isomorphism:

$$\Phi : K^\times/(K^\times)^n \to \mathrm{Hom}(G_K, \mu_n),$$

where $G_K := \mathrm{Gal}(\overline{K}/K)$ is the absolute Galois group of K.

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Introduce Galois cohomology and provide two applications for Prof. Kim's talk.

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Let K be a number field and suppose that $\mu_n \subset K$. Then we have an isomorphism:

$$\Phi : K^\times/(K^\times)^n \to \mathrm{Hom}(G_K, \mu_n),$$

where $G_K := \mathrm{Gal}(\overline{K}/K)$ is the absolute Galois group of K.

Theorem B (Weak Mordell–Weil Theorem)

Let E be an elliptic curve over a number field K. Then $E(K)/nE(K)$ is finite for any $n \geq 2$.

Part I: Galois cohomology

étale cohomology $\longrightarrow$ Galois cohomology $\longrightarrow$ Group cohomology

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Q: What do we expect about cohomology theory?

étale cohomology $\longrightarrow$ Galois cohomology $\longrightarrow$ Group cohomology

Q: What do we expect about cohomology theory?

G a group, M a G-module $\longmapsto$ $\boxed{H^i(G,M)}$ an abelian group.

$\forall$ a short exact sequence of G-modules:

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0,$$

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$\exists$ a long exact sequence of G-modules:

$$0 \longrightarrow H^0(G,L) \longrightarrow H^0(G,M) \longrightarrow H^0(G,N) \longrightarrow$$
$$\longrightarrow H^1(G,L) \longrightarrow H^1(G,M) \longrightarrow H^1(G,N) \longrightarrow$$
$$\longrightarrow H^2(G,L) \longrightarrow H^2(G,M) \longrightarrow H^2(G,N) \longrightarrow$$
$$\longrightarrow H^3(G,L) \longrightarrow \cdots$$

G-modules: abelian groups having an action of a group G.

e.g. μ_n the group of n-th roots of unity, action of $G_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ or $G = \mathrm{Gal}(\mathbf{Q}(\mu_n)/\mathbf{Q})$.

$$\zeta_n \text{ a primitive } n\text{-th root of unity} \longmapsto \langle \zeta_n \rangle \simeq \mu_n.$$

$$\forall \sigma \in G_{\mathbf{Q}}, \quad \sigma(\zeta_n) = \zeta_n^k \quad \text{for some integer } k.$$

The action of $G_{\mathbf{Q}}$ on μ_n factors through G and so $H^i(G_{\mathbf{Q}}, \mu_n) = H^i(G, \mu_n)$.

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> Q: Can we compute $H^i(G_{\mathbf{Q}}, \mu_n)$?

We didn't define these groups yet....

Let $\mathscr{A}$ and $\mathscr{B}$ be two abelian categories. Suppose that $\mathscr{A}$ has enough injectives. Then for a left exact functor $F : \mathscr{A} \to \mathscr{B}$, there is a functor

$$R^i F : \mathscr{A} \to \mathscr{B}$$

such that $\forall$ a short exact sequence in $\mathscr{A}$

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$\exists$ a long exact sequence in $\mathscr{B}$

$$\begin{aligned}
0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) &\longrightarrow \\
\longrightarrow R^1F(A) \longrightarrow R^1F(B) \longrightarrow R^1F(C) &\longrightarrow \\
\longrightarrow R^2F(A) \longrightarrow R^2F(B) \longrightarrow \cdots &
\end{aligned}$$

Construction

Choose an injective resolution of an object $A \in \mathscr{A}$

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow I^3 \longrightarrow \cdots$$

We then obtain a cochain complex

$$0 \longrightarrow F(I^0) \longrightarrow F(I^1) \longrightarrow F(I^2) \longrightarrow F(I^3) \longrightarrow \cdots$$

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This construction does not depend on the choice of a resolution!

Group cohomology

Let Mod(G) be the category of G-modules, or equivalently $\mathbf{Z}[G]$-modules.

Also, let Mod$(\mathbf{Z})$ be the category of abelian groups, or equivalently $\mathbf{Z}$-modules.

Consider the functor $F = (-)^G$: Mod$(G) \to$ Mod$(\mathbf{Z})$.

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Exercise

Prove that F is left exact.

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Definition

For a G-module M, the i-th cohomology group $H^i(G, M)$ is defined as

$$H^i(G, M) := R^i F(M).$$

Real life: How to compute it?

$$H^0(G_{\mathbf{Q}}, \mu_n) = \mu_n^{G_{\mathbf{Q}}} = 1 \text{``}={\text{''}} 0$$

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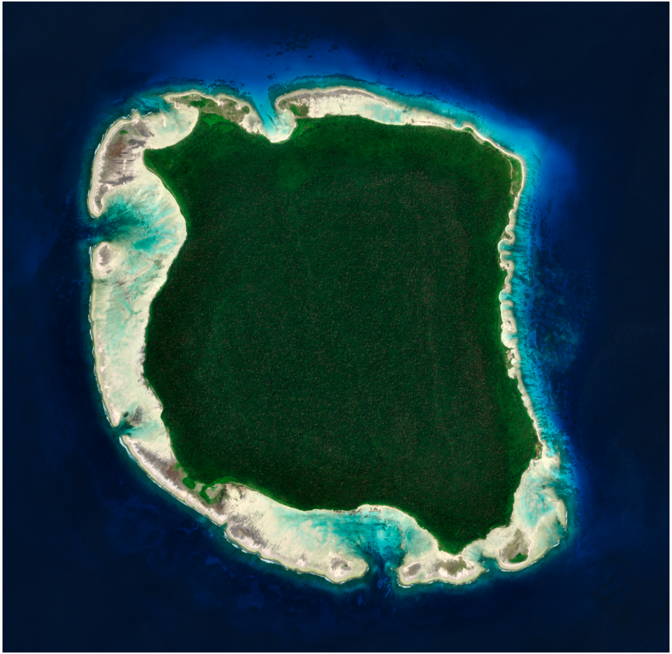
$$H^1(G_{\mathbf{Q}},\mu_n)=? \quad H^2(G_{\mathbf{Q}},\mu_n)=?$$



Another look

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Theorem

Let

$$\cdots \longrightarrow P^3 \longrightarrow P^2 \longrightarrow P^1 \longrightarrow P^0 \longrightarrow \mathbf{Z} \longrightarrow 0$$

be a projective resolution of $\mathbf{Z}$. Then $H^i(G, M)$ is equal to the i-th cohomology of the cochain complex

$$\cdots \longrightarrow \mathrm{Hom}_{\mathbf{Z}[G]}(P^2, M) \longrightarrow \mathrm{Hom}_{\mathbf{Z}[G]}(P^1, M) \longrightarrow \mathrm{Hom}_{\mathbf{Z}[G]}(P^0, M) \longrightarrow 0.$$

Namely, if

$$P^\bullet \to \mathbf{Z} \to 0 \quad \text{and} \quad 0 \to M \to I^\bullet$$

are two (projective and injective) resolutions, then the i-th cohomologies of the following two cochain complexes are equal:

$$\mathsf{Hom}_{\mathbf{Z}[G]}(P^\bullet, M) \quad \text{and} \quad \mathsf{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, I^\bullet).$$

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There is a very good free resolution of $\mathbf{Z}$!

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Exercise

Study the standard complex or bar resolution.

Alternative definition of H^1

Let G be a finite group acting on an abelian group M. A crossed homomorphism is a map $f : G \to M$ such that

$$f(\sigma\tau) = f(\sigma) + \sigma \cdot f(\tau) \quad \text{for all } \sigma, \tau \in G$$

and it is said to be principal if there is an element $m \in M$ such that

$$f(\sigma) = \sigma \cdot m - m \quad \text{for all } \sigma in G.$$

We then have

$$H^1(G, M) = \frac{\text{crossed homomorphisms}}{\text{principal crossed homomorphisms}}.$$

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By definition, if G acts trivially on M, then we have

$$\boxed{H^1(G, M) = \text{Hom}(G, M)}$$

Change the group

Let $\lambda : H \to G$ be a group homomorphism. Then λ gives rise to an exact functor

$$\Phi_\lambda : \mathsf{Mod}(G) \to \mathsf{Mod}(H)$$

because every G-module can be considered as a H-module via λ. In particular, if H is a subgroup of G, then we have

$$\mathsf{res}_H^G : H^i(G, M) \to H^i(H, M).$$

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$$\mathsf{res}^G_H : H^i(G, M) \to H^i(H, M).$$

Also, if N is a normal subgroup of G, then we may take $(G, H) = (G/N, G)$ (and λ is the quotient map), and hence we obtain

$$\mathsf{inf}^{G/N}_G : H^i(G/N, M^N) \to H^i(G, M^N) \to H^i(G, M).$$

Inflation and Restriction

Theorem

Let G be a group and N a normal subgroup. Then for $M \in \text{Mod}(G)$ we have an exact sequence

$$0 \longrightarrow H^1(G/N, M^N) \xrightarrow{\text{inf}} H^1(G, M) \xrightarrow{\text{res}} H^1(N, M)^{G/N} \longrightarrow H^2(G/N, M^N) \xrightarrow{\text{inf}} H^2(G, M).$$

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Exercise

Study the Grothendieck spectral sequence.

Part II: Applications

Hilbert's Theorem 90

Theorem (Kummer, Hilbert, Noether)

Let L/K be a finite Galois extension with Galois group G. Then $H^1(G, L^\times) = 0$.

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Proof. Let $f : G \to L^\times$ be a crossed homomorphism. In multiplicative notation, this means that for any $\sigma, \tau \in G$, we have $f(\sigma\tau) = f(\sigma)\sigma(f(\tau))$ or equivalently

$$\boxed{\sigma(f(\tau)) = f(\sigma)^{-1} f(\sigma\tau)},$$

and we have to find $\boxed{m \in L^\times}$ such that $f(\sigma) = \sigma(m)/m$ for all $\sigma \in G$.

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Lemma (Dedekind)

Let L/K be a finite Galois extension. Then distinct elements of $\mathrm{Gal}(L/K)$ are linear independent over L.

As $f(\tau) \in L^\times$ is nonzero, the above lemma implies that

$$\sum_{\tau \in G} f(\tau) \cdot \tau : L \to L$$

is not a zero map, i.e., there exists an $\alpha \in L$ such that

$$\beta := \sum_{\tau \in G} f(\tau) \cdot \tau(\alpha) \neq 0.$$

But then, for $\sigma \in G$, we have

$$\begin{aligned}\sigma(\beta) &= \sum_{\tau \in G} \sigma(f(\tau)) \cdot \sigma\tau(\alpha) \\ &= \sum_{\tau \in G} f(\sigma)^{-1} f(\sigma\tau) \cdot \sigma\tau(\alpha) \\ &= f(\sigma)^{-1} \sum_{\tau \in G} f(\sigma\tau) \cdot \sigma\tau(\alpha) = f(\sigma)^{-1}\beta\end{aligned}$$

as τ runs over G, so also does $\sigma\tau$. Thus, we have $f(\sigma) = \beta/\sigma(\beta) = \sigma(\beta^{-1})/\beta^{-1}$. $\square$

Infinite Galois theory

Let L/K be a Galois extension with infinite Galois group G and M a G-module. The group G has natural profinite topology, i.e., basic open sets of G are those subgroups $H < G$ which have finite index in G. We then define the cohomology groups of G with coefficients in A as

$$H^i(G, M) := \varinjlim H^i(G/H, M^H),$$

where H runs through all open subgroups of G. (Use the inflation maps!)

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$$H^i(G, M) := \varinjlim H^i(G/H, M^H),$$

where H runs through all open subgroups of G. (Use the inflation maps!)

Theorem

Let L/K be an infinite Galois extension with Galois group G. Then $H^1(G, L^\times) = 0$.

Proof.

Exercise! □

Classification of quadratic / cubic extensions

Question

Can we classify all the quadratic extensions of $\mathbb{Q}$?

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Kummer theory

Suppose that K is a number field containing a primitive n-th root of unity ζ_n, or equivalently $\mu_n \subset K$ for a given integer $n \geq 2$. Then we can easily classify abelian extensions of exponent n in terms of some data related to $K^\times$ (cf. CFT).

More precisely, for any $a \in K^\times$, the field $L = K(\sqrt[n]{a})$ is the splitting field of $f(x) = x^n - a$ over K; the notation $\sqrt[n]{a}$ denotes a particular primitive n-th root of a, but it does not matter which root we pick because $\mu_n \subset K$ (and so all the n-th roots of a are of the form $\zeta_n^k \sqrt[n]{a}$). Note that L is a Galois extension of K, and $\mathrm{Gal}(L/K)$ is cyclic as we have an injective homomorphism:

$$\begin{array}{ccc} \mathrm{Gal}(L/K) & \hookrightarrow & \mu_n \simeq \mathbf{Z}/n\mathbf{Z} \\ \sigma & \longmapsto & \frac{\sigma(\sqrt[n]{a})}{\sqrt[n]{a}} \end{array}$$

This homomorphism is an isomorphism if and only if $x^n - a$ is irreducible.

Lemma

Let L/K be a cyclic field extension of degree n with Galois group $\langle\sigma\rangle$ and suppose that L contains a primitive n-th root of unity ζ_n. Then $\sigma(\alpha) = \zeta_n \alpha$ for some $\alpha \in L$.

Proof.

The automorphism σ is a linear transformation of L with characteristic polynomial $x^n - 1$; by the above lemma by Dedekind it must be its minimal polynomial, since $\{1, \sigma, \sigma^2, \ldots, \sigma^{n-1}\}$ is linearly independent. Thus, ζ_n is an eigenvalue of σ. $\square$

Classification of cyclic extensions

Theorem (classification)

Let K be a number field containing a primitive n-th root of unity ζ_n. If L/K is a cyclic extension of degree n, then $L = K(\sqrt[n]{a})$ for some $a \in K^\times$.

Proof.

By the above lemma, there is an element $\alpha \in L$ for which $\sigma(\alpha) = \zeta_n \alpha$. We have

$$\sigma(\alpha^n) = \sigma(\alpha)^n = (\zeta_n \alpha)^n = \alpha^n,$$

thus $a = \alpha^n$ is invariant under the action of $\langle \sigma \rangle = \mathrm{Gal}(L/K)$ and thus lies in K. Moreover, the orbit $\{\alpha, \zeta\alpha, \ldots, \zeta^{n-1}\alpha\}$ of α under the action of $\mathrm{Gal}(L/K)$ has order n, so

$$L = K(\alpha) = K(\sqrt[n]{a}).$$

□

Kummer pairing

Definition

Let K be a number field and assume that $\zeta_n \in K$. The Kummer pairing is the map

$$\langle -,-\rangle : \mathrm{Gal}(\overline{K}/K) \times K^\times \longrightarrow \langle \zeta_n \rangle = \mu_n$$

$$\langle \sigma, a \rangle \longmapsto \frac{\sigma(\sqrt[n]{a})}{\sqrt[n]{a}}$$

which is well-defined. Indeed, if α and β are two n-th roots of a, then $(\alpha/\beta)^n = 1$ and so $\alpha/\beta \in \langle \zeta_n \rangle \subset K$ is fixed by σ. Thus,

$$\sigma(\beta)/\beta = \sigma(\beta)/\beta \cdot \sigma(\alpha/\beta)/(\alpha/\beta) = \sigma(\alpha)/\alpha$$

and the value of $\langle \sigma, a \rangle$ does not depend on the choice of $\sqrt[n]{a}$.

First Proof of Theorem A

From the Kummer pairing, we have a natural map sending $a \in K^\times$ to $(\sigma \mapsto \langle \sigma, a \rangle)$:

$$\Phi : K^\times \to \mathrm{Hom}(G_K, \mu_n)$$

It suffices to show that $\ker(\Phi) = (K^\times)^n$ and Φ is surjective.

1) For each $a \in K^\times \setminus (K^\times)^n$, if we pick an n-th root $\alpha \in \overline{K}$, then the extension $K(\alpha)/K$ is non-trivial and some $\sigma \in G_K$ must act nontrivially on α. For this σ, we have $\langle \sigma, a \rangle \neq 1$ and so $a \notin \ker(\Phi)$. Note that $(K^\times)^n \subset \ker(\Phi)$ is obvious.

2) Surjectivity is an exercise. Use the classification theorem.

Another proof of Theorem A

The multiplicative group $\overline{K}^{\times}$ is a G_K-module and there is an exact sequence of G_K-modules:

$$0 \longrightarrow \mu_n \longrightarrow \overline{K}^{\times} \xrightarrow{(-)^n} \overline{K}^{\times} \longrightarrow 0.$$

Taking a long exact sequence of cohomology yields:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mu_n^{G_K} & \longrightarrow & (\overline{K}^{\times})^{G_K} & \xrightarrow{(-)^n} & (\overline{K}^{\times})^{G_K} & \longrightarrow & H^1(G_K, \mu_n) & \longrightarrow & H^1(G_K, \overline{K}^{\times}) \\
& \| (a) & & \| (b) & & \| (c) & & \| (d) & & \| (e) \\
& \mu_n & \longrightarrow & K^{\times} & \xrightarrow{(-)^n} & K^{\times} & \longrightarrow & \mathrm{Hom}(G_K, \mu_n) & \longrightarrow & 0
\end{array}$$

Another proof of Theorem A

The multiplicative group $\overline{K}^\times$ is a G_K-module and there is an exact sequence of G_K-modules:

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\| (a) & & \| (b) & & \| (c) & & \| (d) & & \| (e) \\
\mu_n & \longrightarrow & K^\times & \xrightarrow{(-)^n} & K^\times & \longrightarrow & \mathrm{Hom}(G_K, \mu_n) & \longrightarrow & 0
\end{array}$$

Why? $(a), (d)$: Note that we assume that $\mu_n \subset K$, and so G_K acts trivially on μ_n.

$(b), (c)$: Galois theory. (e) : Hilbert's theorem 90. □

Elliptic curves

Note that the group scheme $\mathbf{G}_m$ is used in the previous discussion. A bit more specifically,

$$\mathbf{G}_m(L) = L^\times \quad \text{and} \quad \mathbf{G}_m(\overline{K}) = \overline{K}^\times.$$

Elliptic curves

Note that the group scheme $\mathbf{G}_m$ is used in the previous discussion. A bit more specifically,

$$\mathbf{G}_m(L) = L^\times \quad \text{and} \quad \mathbf{G}_m(\overline{K}) = \overline{K}^\times.$$

Other type of group schemes can be used in a similar manner: Let E be an elliptic curve over a number field K. As above, there is an exact sequence of G_K-modules:

$$0 \longrightarrow E[n] \longrightarrow E(\overline{K}) \xrightarrow{\times n} E(\overline{K}) \longrightarrow 0.$$

Here, $E[n] := \{P \in E(\overline{K}) : nP = 0\}$ is the group of n-torsion points.

Taking a long exact sequence of cohomology gives rise to:

$$0 \longrightarrow E[n]^{G_K} \longrightarrow E(\overline{K})^{G_K} \xrightarrow{\times n} E(\overline{K})^{G_K} = E(K)$$

$$\longrightarrow H^1(G_K, E[n]) \longrightarrow H^1(G_K, E(\overline{K})) \xrightarrow{\times n} H^1(G_K, E(\overline{K})).$$

Taking a long exact sequence of cohomology gives rise to:

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$$\longrightarrow H^1(G_K, E[n]) \longrightarrow H^1(G_K, E(\overline{K})) \xrightarrow{\times n} H^1(G_K, E(\overline{K})).$$

Thus, we obtain a short exact sequence:

$$0 \longrightarrow E(K)/nE(K) \longrightarrow \boxed{H^1(G_K, E[n])} \longrightarrow H^1(G_K, E(\overline{K}))[n] \longrightarrow 0.$$

If $H^1(G_K, E[n])$ were **finite**, we would be very happy. But unfortunately, it is NOT...

Local picture

For a prime v, we fix an extension of v to $\overline{K}$. We then have a commutative diagram:

$$\begin{array}{ccc} \overline{K} & \hookrightarrow & \overline{K}_v \\ | & & | \\ K & \overset{\iota_v}{\hookrightarrow} & K_v \end{array}$$

and so a decomposition group $G_v = \mathrm{Gal}(\overline{K}_v/K_v) \subset G_K$. Now G_v acts on $E(\overline{K}_v)$ and similarly as above we get:

$$0 \longrightarrow E(K_v)/nE(K_v) \longrightarrow H^1(G_v, E[n]) \longrightarrow H^1(G_v, E(\overline{K}_v))[n] \longrightarrow 0.$$

Via the maps $E(K) \hookrightarrow E(K_v)$ and $G_v \subset G_K$, we get commutative short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E(K)/nE(K) & \xrightarrow{\iota} & H^1(G_K, E[n]) & \longrightarrow & H^1(G_K, E(\overline{K}))[n] & \longrightarrow & 0,\\
 & & \downarrow & \searrow^{f_v} & \downarrow{\scriptstyle \mathrm{res}} & \searrow^{g_v} & \downarrow{\scriptstyle \mathrm{res}} & & \\
0 & \longrightarrow & E(K_v)/nE(K_v) & \longrightarrow & \boxed{H^1(G_v, E[n])} & \longrightarrow & H^1(G_v, E(\overline{K}_v))[n] & \longrightarrow & 0.
\end{array}$$

If E has good reduction at v and $v \nmid n$, then the action of G_v on $E[n]$ is unramified (Néron–Ogg–Shafarevich criterion) so it factors through the quotient $G_v/I_v \simeq \langle \mathrm{Frob}_v \rangle \simeq \widehat{\mathbf{Z}}$. Furthermore, if $E[n] = E(K_v)[n]$, then

$$H^1(G_v, E[n]) = \mathrm{Hom}(\widehat{\mathbf{Z}}, (\mathbf{Z}/n\mathbf{Z})^2) \simeq (\mathbf{Z}/n\mathbf{Z})^2$$

is obviously finite. If f_v were **injective**, we would be very happy. But it is NOT...

Selmer groups (and TS)

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & E(K)/nE(K) & \xrightarrow{\iota} & H^1(G_K, E[n]) & \longrightarrow & H^1(G_K, E(\overline{K}))[n] & \longrightarrow & 0, \\
 & & \downarrow & \overset{f_v}{\searrow} & \downarrow & \overset{g_v}{\searrow} & \downarrow & & \\
0 & \longrightarrow & E(K_v)/nE(K_v) & \longrightarrow & \boxed{H^1(G_v, E[n])} & \longrightarrow & H^1(G_v, E(\overline{K}_v))[n] & \longrightarrow & 0.
\end{array}
$$

However, since the diagram is commutative any element in the image of ι maps to 0 by g_v.
This motivates the following definition...

Selmer groups (and TS)

Definition

The n-Selmer group of E/K is the group

$$\mathrm{Sel}^{(n)}(E/K) := \ker\left(H^1(G_K, E[n]) \to \prod_{\text{all } v} H^1(G_v, E(\overline{K}_v))\right)$$

and the Tate–Shafarevich group of E/K is the group

$$\text{Ш}(E/K) := \ker\left(H^1(G_K, E(\overline{K})) \to \prod_{\text{all } v} H^1(G_v, E(\overline{K}_v))\right).$$

Proof of Theorem B

From the discussion above, we obtain a short exact sequence:

$$0 \longrightarrow E(K)/nE(K) \longrightarrow \boxed{\mathrm{Sel}^{(n)}(E/K)} \longrightarrow \text{Ш}(E/K)[n] \longrightarrow 0.$$

Theorem C

The group $\mathrm{Sel}^{(n)}(E/K)$ is finite. Hence $E(K)/nE(K)$ and $\text{Ш}(E/K)[n]$ are also finite.

Conjecture

The group $\text{Ш}(E/K)$ is finite.

Sketch of the proof

First, we may consider the finite extension $L = K(E[n])$ of K. Then it is not hard to prove that $E(K)/nE(K)$ is finite if $E(L)/nE(L)$ is finite.

Then we construct the same exact sequences for L instead of K. Note that $H^1(G_L, E[n]) = \mathrm{Hom}(G_L, E[n])$ as G_L acts trivally on $E[n]$. It turns out that an element of $\mathrm{Sel}^{(n)}(E/L)$ (as a subgroup of $\mathrm{Hom}(G_L, E[n])$) is a special map from G_L to $E[n]$. Furthermore, such a map corresponds to a finite extension of exponent n of L unramified outside a finite set S.

Finally, the number of such “unramified” extensions is finite.

Sketch of the proof

First, we may consider the finite extension $L = K(E[n])$ of K. Then it is not hard to prove that $E(K)/nE(K)$ is finite if $E(L)/nE(L)$ is finite.

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Remark

We can directly prove that $\mathrm{Sel}^{(n)}(E/K)$ is finite by a similar argument.

References

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Thank you very much
for your attention!