Lectures on quantum groups

1. Intro. to quantum groups and their representations
2. Crystal bases for integrable highest weight modules
3. Combinatorial realization of crystals.

References

Bases cristallines des groupes quantiques (Kashiwara)

Lectures on quantum groups (Jantzen)

Introduction to quantum groups. (Lusztig)

What is quantum group ?

$\mathfrak{g}$: semisimple Lie alg $/\mathbb{C}$ $\subset$ $U(\mathfrak{g})$: universal enveloping alg. $/\mathbb{C}$

(non-associative)

$U_q(\mathfrak{g})$: q-analogue of $U(\mathfrak{g})$

(Drinfeld - Jimbo)

assoc. alg $/\ \mathbb{C}(q)$

(q : indeterminate)

Representation of $U_q(\mathfrak{g}) \xrightarrow[q \to 1]{}$ Representations of $U(\mathfrak{g})$ or $\mathfrak{g}$

$\exists$ info (structure, basis, etc) which can't be seen when $q = 1$.

V : finite-dim repn of $\mathfrak{g}$

$\rightsquigarrow$ semisimple = a direct sum of simple (or irreducible) repn's

<u>Question</u> How to decompose V ?

$\exists$ beautiful combinatorial structure of $U_q(\mathfrak{g})$-mod's when $q = 0$

(crystal base by Kashiwara)

A theory of crystal base provides a combinatorial solution.

Lecture 1. Quantized enveloping algebra

q-analogue of the universal enveloping alg.

We assume the following data

- I : index set (finite)
- P : free abelian group
- $\{\alpha_i \mid i \in I\} \subset P$ the set of simple roots (linearly independent)
- $\{h_i \mid i \in I\} \subset P^\vee = \mathrm{Hom}(P, \mathbb{Z})$: the set of simple coroots.
- $(\cdot\,,\cdot) : P \times P \longrightarrow \mathbb{Q}$ symm bilinear form s.t.

1) $(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0} \quad (i \in I)$

2) $(\alpha_i, \alpha_j) \leq 0 \quad (i, j \in I,\ i \neq j)$

3) $\langle h_i, \lambda \rangle = \dfrac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \quad (i \in I,\ \lambda \in P)$

Rmk $A = (a_{ij})_{i,j \in I} \qquad a_{ij} = \langle h_i, \alpha_j \rangle$

A : a generalized Cartan matrix

A : finite type if A : positive definite

affine if A : positive semidefinite & $\det A = 0$

$\mathfrak{g}$: a Kac-Moody alg assoc. to $A \rightsquigarrow U(\mathfrak{g})$

- q : formal variable.

- $K = \Bbbk(q)$ ($\Bbbk$: a field of $ch = 0$)

- $q_i = q^{\frac{(\alpha_i, \alpha_i)}{2}}$ ($i \in I$)

$$[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}} = q_i^{n-1} + q_i^{n-3} + \cdots + q_i^{-n+1} \qquad (\text{q-integer})$$

$$[n]_i! = [n]_i [n-1]_i \cdots [2]_i [1]_i .$$

$$\begin{bmatrix} n \\ m \end{bmatrix}_i = \frac{[n]_i!}{[m]_i! [n-m]_i!} \qquad (\text{q-binomial coeff})$$

Def. $U_q(\mathfrak{g})$ = the assoc. $\mathbb{K}$-alg. with 1

- generators : $e_i, f_i, q^h \quad (h \in P^\vee, i \in I)$
- relations :

$$q^h q^{h'} = q^{h'} q^h = q^{h+h'}, \quad q^0 = 1$$

$$q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i \qquad q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i$$

$$e_i f_j - f_j e_i = \delta_{ij} \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}} \quad \text{where } t_i = q^{\frac{(\alpha_i, \alpha_i)}{2} h_i}$$

$$\sum_{k=0}^{c_{ij}} (-1)^k e_i^{(k)} e_j e_i^{(c_{ij}-k)} = \sum_{k=0}^{c_{ij}} (-1)^k f_i^{(k)} f_j f_i^{(c_{ij}-k)} = 0$$

where $c_{ij} = 1 - a_{ij}$, $\quad x_i^{(k)} = \dfrac{x_i^k}{[k]_i!}$

$U_q(\mathfrak{g})$: the quantized enveloping algebra (of $\mathfrak{g}$)

(Drinfeld - Jimbo)

Rmk

① $A = (2)$ $\quad U_q(\mathfrak{sl}_2) = \langle e, f, t^{\pm 1} \rangle$

$$t e t^{-1} = q^2 e \qquad t f t^{-1} = q^{-2} f \qquad ef - fe = \frac{t - t^{-1}}{q - q^{-1}}$$

② $t_i e_j t_i^{-1} = q_i^{\langle h_i, \alpha_j \rangle} e_j \left(= q^{(\alpha_i, \alpha_j)} e_j \right)$

$$e_i \longrightarrow e \qquad f_i \longrightarrow f \qquad t_i \longrightarrow t \qquad q_i \longrightarrow q$$

$$\langle e_i, f_i, t_i^{\pm 1} \rangle / k(q_i) \cong U_q(\mathfrak{sl}_2)$$

<u>Example</u> $\mathfrak{g} = \mathfrak{gl}_3$ type A_2

$$P = \mathbb{Z}\delta_1 \oplus \mathbb{Z}\delta_2 \oplus \mathbb{Z}\delta_3 \qquad (\delta_i, \delta_j) = \delta_{ij}$$

$$\alpha_1 = \delta_1 - \delta_2, \quad \alpha_2 = \delta_2 - \delta_3$$

$$P^\vee = \mathbb{Z}\delta_1^\vee \oplus \mathbb{Z}\delta_2^\vee \oplus \mathbb{Z}\delta_3^\vee \qquad \langle \delta_j^\vee, \delta_i \rangle = \delta_{ij}$$

$$U_q(\mathfrak{gl}_3) = \langle e_i, f_i \ (i=1,2) \quad q^{\pm\delta_j^\vee} \ (j=1,2,3) \rangle$$

$$x_1^{(2)} x_2 - x_1 x_2 x_1 + x_2 x_1^{(2)} = 0 \qquad (x = e, f)$$

$$\vdots$$

- (Triangular decomposition)

$$U_q^+(\mathfrak{g}) := \langle e_i \mid i \in I \rangle, \quad U_q^-(\mathfrak{g}) := \langle f_i \mid i \in I \rangle$$

$$U_q^0(\mathfrak{g}) := \langle q^h \mid h \in P^\vee \rangle$$

Then

$$\begin{array}{ccc} U_q^-(\mathfrak{g}) \underset{K}{\otimes} U_q^0(\mathfrak{g}) \underset{K}{\otimes} U_q^+(\mathfrak{g}) & \xrightarrow{\cong} & U_q(\mathfrak{g}) \\ x_- \otimes x_0 \otimes x_+ & \longmapsto & x_- x_0 x_+ \end{array} \qquad K\text{-linear iso}$$

- (Weight space decomposition)

$$U_q(\mathfrak{g}) = \bigoplus_{\xi \in P} U_q(\mathfrak{g})_\xi \qquad U_q(\mathfrak{g})_\xi = \{ x \mid q^h x q^{-h} = q^{\langle h, \xi \rangle} x \ (h \in P^\vee) \}$$

$$Q = \sum_{i \in I} \mathbb{Z}\alpha_i \qquad Q_+ = \sum \mathbb{Z}_+ \alpha_i , \quad Q_- = -Q_+$$

$$U_q^{\pm}(\mathfrak{g}) = \bigoplus_{Q_\pm} U_q(\mathfrak{g})_\beta$$

comult antipode counit.

- $U_q(\mathfrak{g})$ is a Hopf algebra with Δ, S, ε

$$\Delta(e_i) = 1 \otimes e_i + e_i \otimes t_i^{-1}$$

$$\Delta(f_i) = f_i \otimes 1 + t_i \otimes f_i$$

$$\Delta(q^h) = q^h \otimes q^h$$

$$S(e_i) = -e_i t_i \qquad S(f_i) = -t_i^{-1} f_i \qquad S(q^h) = q^{-h}$$

$$\varepsilon(e_i) = \varepsilon(f_i) = \varepsilon(q^h) = 0 \qquad \varepsilon(1) = 1$$

Representations of $U_q(\mathfrak{g})$

- $\lambda \in P_+$ $\left(\text{i.e. } \langle h_i, \lambda \rangle \in \mathbb{Z}_{\geq 0} \ (i \in I) \right)$

  $V(\lambda)$: a left $U_q(\mathfrak{g})$-module gen. by v_λ

  subject to the following relations

  $$q^h \cdot v_\lambda = q^{\langle h, \lambda \rangle} v_\lambda \qquad e_i v_\lambda = 0 \qquad f_i^{\langle h_i, \lambda \rangle + 1} v_\lambda = 0$$

  $$(h \in P^\vee, \ i \in I)$$

  that is, $V(\lambda) :=$

  $$\frac{U_q(\mathfrak{g})}{\sum_i U_q(\mathfrak{g}) e_i + \sum_i U_q(\mathfrak{g}) f_i^{\langle h_i, \lambda \rangle + 1} + \sum_h U_q(\mathfrak{g}) (q^h - q^{\langle h, \lambda \rangle})} \ni v_\lambda = \bar{1}$$

By triangular decomposition,

$$V(\lambda) = U_q^-(\mathfrak{g})\, v_\lambda = \bigoplus_{\mu \leq \lambda} V(\lambda)_\mu$$

where $V(\lambda)_\mu = \{ v \mid q^h v = q^{\langle h, \mu \rangle} v \}$

<u>Thm</u> $V(\lambda)$ is irreducible.

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pf.) Use the classical limit of $V(\lambda)$

$A = k[q, q^{-1}]$

$U_q(\mathfrak{g})_{\mathbb{A}}$ = the A-subalg. gen. by $e_i^{(n)}, f_i^{(n)}, q^h, \dfrac{q^h - q^{-h}}{q - q^{-1}}$

$$U_q(\mathfrak{g})_{\mathbb{Z}} \cong U_q^-(\mathfrak{g})_{\mathbb{Z}} \otimes U_q^0(\mathfrak{g})_{\mathbb{Z}} \otimes U_q^+(\mathfrak{g})_{\mathbb{Z}} \quad \text{as } A\text{-modules.}$$

$$V(\lambda)_{\mathbb{Z}} := U_q(\mathfrak{g})_{\mathbb{Z}} \cdot v_\lambda = U_q^-(\mathfrak{g})_{\mathbb{Z}}\, v_\lambda \quad \circlearrowleft \; U_q(\mathfrak{g})_{\mathbb{Z}}$$

$$\overline{V(\lambda)} := V(\lambda)_{\mathbb{Z}} \otimes_A \Bbbk \qquad \text{where } \Bbbk : \text{an } A\text{-module w/ } f(q)\cdot 1 := f(1)$$

$$\overline{e_i},\ \overline{f_i},\ \overline{h} \in \mathrm{End}_{\Bbbk}\left(\overline{V(\lambda)}\right) \qquad \text{induced from } e_i,\ f_i,\ \frac{q^h - q^{-h}}{q - q^{-1}}$$

Then $\exists\ U(\mathfrak{g}) \longrightarrow \mathrm{End}_{\Bbbk}(\overline{V(\lambda)})$ $\Bbbk$-alg. homo.

$\overline{V(\lambda)}$: a $U(\mathfrak{g})$-module with h.w. vector $v_\lambda \otimes 1$.

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Note ① $\dim_{\bar{k}} \overline{V(\lambda)}_\mu = \dim_K V(\lambda)_\mu = \operatorname{rank}_A (V(\lambda)_{\mathbb{Z}})_\mu$

② $\overline{V(\lambda)}$: irr. $\mathcal{U}(\mathfrak{g})$-module

$0 \neq W \subset V(\lambda)$: $U_q(\mathfrak{g})$-submodule. $\left(\Rightarrow W = \bigoplus_{\mu \in P} W_\mu \right)$

ν : maximal weight of W & $0 \neq w \in W_\nu$ $\quad (\nu \leq \lambda)$

May assume $w \in W \cap V(\lambda)_{\mathbb{Z}}$

$W'_{\mathbb{Z}} := U_q(\mathfrak{g})_{\mathbb{Z}} \, w_\nu = U_q^-(\mathfrak{g})_{\mathbb{Z}} \cdot w_\nu$

$\overline{W'_{\mathbb{Z}}}$: $\underbrace{a}_{\text{non-zero}}$ $\mathcal{U}(\mathfrak{g})$-submodule of $\overline{V(\lambda)}$

$$\Rightarrow \overline{W'_{\mathbb{Z}}} = \overline{V(\lambda)} \Rightarrow v = \lambda$$

$$V(\lambda)_\lambda = K v_\lambda \Rightarrow w = c v_\lambda \Rightarrow W = V(\lambda)$$

$\therefore$ $V(\lambda)$: irreducible

$\boxtimes$

<u>Rmk</u> $\overline{V(\lambda)}$: the classical limit of $V(\lambda)$

$$\mathrm{ch}_K V(\lambda) = \mathrm{ch}_{\mathbb{k}} \overline{V(\lambda)} = \frac{\sum_W (-1)^{\ell(w)} e^{w(\lambda+\rho)-\rho}}{\prod_{\Delta^+} (1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}$$

- $\mathcal{O}_{\mathrm{int}}$: the category of $U_q(\mathfrak{g})$-modules M such that

1) $M = \bigoplus_{\lambda \in P} M_\lambda \qquad \dim M_\lambda < \infty$

2) $\underline{\mathrm{wt}(M)} \subset (\lambda_1 - Q_+) \cup \cdots \cup (\lambda_r - Q_+)$ for some $\lambda_1, \cdots, \lambda_r$

set of weights

3) e_i, f_i : locally nilpotent for $i \in I$

Rmk

- $\mathcal{O}_{\mathrm{int}}$: closed under submodule, quotient & $\otimes$
- $V(\lambda) \in \mathcal{O}_{\mathrm{int}} \quad (\lambda \in P_+)$

• $L \in \mathcal{O}_{int}$: simple $\Rightarrow$ $L = V(\lambda)$ for some $\lambda \in P_+$

($\because$ $\exists$ $v \in L$ of maximal wt. λ by 2)

$\lambda \in P_+$ by 3)

• Every highest weight module in $\mathcal{O}_{int}$ is irreducible

($\because$ V : a h.w. module with h.w. vector of wt λ

$\overline{V}$: the classical limit (h.w. module)

$\Rightarrow$ $\overline{V} \cong \overline{V(\lambda)}$ for some $\lambda \in P_+$ $\Rightarrow$ $V \cong V(\lambda)$

$\because$ a quotient of a Verma

- $V(-\lambda)$: an irr. lowest weight modules for $\lambda \in P_+$

  $\mathcal{O}^*_{\mathrm{int}}$: the category with irreducibles $V(-\lambda)$'s

$$\mathcal{O}_{\mathrm{int}} \xrightarrow{\cong} \mathcal{O}^*_{\mathrm{int}}$$

$$M = \bigoplus_\lambda M_\lambda \longmapsto M^* = \bigoplus_\lambda \overset{\mathrm{Hom}_K(M_\lambda, K)}{\overset{\shortparallel}{M_\lambda^*}}$$

$(x \cdot f)(u) := f(S(x)u)$

$$\bigoplus_\mu N_\mu^* = N' \longmapsfrom N = \bigoplus_\mu N_\mu$$

$(x \cdot f)(u) = f(S^{-1}(x)u)$

Thm $\mathcal{O}_{int}$: semisimple

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pf.) $V \in \mathcal{O}_{int}$

$v \in V$: a maximal vector of weight λ.

$L := U_q(\mathfrak{g})\, v \cong V(\lambda)$

$v^* \in V^*$ s.t. $v^*(v) = 1$ $\quad v^*|_{V_\mu} = 0$ for $\mu \neq \lambda$

$\Rightarrow$ v^* : a minimal vector of wt $-\lambda$

$\Rightarrow$ $\bar{L} := U_q(\mathfrak{g})\, v^* \cong V(-\lambda)$

$$0 \longrightarrow \bar{L} \longrightarrow V^* \quad \Rightarrow \quad (V^*)' \xrightarrow{\varphi} (\bar{L})' \longrightarrow 0$$

$$(V^*)' \cong V, \quad (\bar{L})' \cong V(\lambda) \cong L$$

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] & V \arrow[l, bend left, "\varphi"] \arrow[r] & V/L \arrow[r] & 0 \end{tikzcd} \quad : \text{split.}$$

$$V \cong L \oplus V/L$$

If V : finitely generated by $F \subset V$, then use induction on $\dim F$ to show that V : semisimple.

In general, $V = \sum_{W : \text{f.g.}} W$ $\quad \therefore V$: semisimple

Rmk $V \in \mathcal{O}_{\text{int}}$

$$V \cong \bigoplus_{\lambda \in P_+} V(\lambda)^{\oplus m_\lambda} \quad \text{for some } m_\lambda \in \mathbb{Z}_{\geq 0}$$

In particular, for $\mu, \nu \in P_+$

$$V(\mu) \otimes V(\nu) \cong \bigoplus_{\lambda} V(\lambda)^{\oplus c_{\mu\nu}^{\lambda}} \quad \text{for some } c_{\mu\nu}^{\lambda}$$

By using the Weyl-Kac char. formula, one may have a formula for $c_{\mu\nu}^{\lambda}$. (Steinberg formula), which is not efficient to compute in general.

Repn's of $U_q(sl_2)$

$m \in \mathbb{Z}_{\geq 0}$

$V(m)$: irr. h.w. $U_q(sl_2)$-module with h.w. m

$$V(m) = \bigoplus_{k=0}^{m} f^{(k)} v_m \qquad \text{where} \quad v_m : \text{h.w. vector}$$

$$f \cdot f^{(k)} v_m = [k+1] \, f^{(k+1)} v_m$$

$$\underline{e \cdot f^{(k)}} \, v_m = \left(\underline{f^{(k)} e + f^{(k-1)} \{ q^{1-k} t \}} \right) v_m$$

$$= [m-k+1] \, f^{(k-1)} v_m$$

$V(m) \otimes V(n)$: semisimple

$$= V(m+n) \oplus V(m+n-2) \oplus \cdots \oplus \underset{\max\{m,n\} - \min\{m,n\}}{\underset{\shortparallel}{V(\ell)}}$$

<u>Example</u> (Tensor product decomposition)

$V(2) \otimes V(1) \cong V(3) \oplus V(1)$: K-span of $\underline{f^{(k)} v_2 \otimes f^{(\ell)} v_1}$

$k = 0, 1, 2 \quad \ell = 0, 1$

$w_3 = v_2 \otimes v_1$: h.w. vector of wt 3.

$$V(3) \cong U_q(sl_2)\, w_3 = \bigoplus_{n=0}^{3} K f^{(n)} w_3$$

$w_1 = a\, v_2 \otimes f v_1 + b\, f v_2 \otimes v_1$: h. w. vector of wt 1.

$$e\, w_1 = e\,(a v_2 \otimes f v_1) + e\,(b\, f_2 v_2 \otimes v_1)$$

$$= a\,(v_2 \otimes v_1) + b\, q^{-1}[2]\; v_2 \otimes v_1 = 0$$

$$a = -b\, q^{-1}[2] = -\underset{\substack{\shortparallel \\ q^2}}{\underline{b}}\,(1+q^{-2})$$

$$w_1 = (1+q^2)\, v_2 \otimes f v_1 - q^2 f v_2 \otimes v_1$$

$$V(1) \cong U_q(sl_2)\, w_1 = K w_1 \oplus K f w_1$$

$$f^n w_3 = \underline{\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{-k(n-k)} f^{n-k} t^k \otimes f^k} \cdot (v_2 \otimes v_1) \qquad n = 0, 1, 2, 3$$

$$f^{(n)} w_3 = \sum_{k=0}^{n} q^{-k(n-k)} f^{(n-k)} t^k \otimes f^{(k)} . (v_2 \otimes v_1)$$

$$= \sum_{k=0}^{n} \underbrace{q^{-k(n-k)} q^{2k}}_{q^{k^2+2k-kn}} f^{(n-k)} v_2 \otimes f^{(k)} v_1$$

$$\equiv \begin{cases} f^{(n)} v_2 \otimes v_1 & (n = 0, 1, 2) \\ f^{(2)} v_2 \otimes f v_1 & (n = 3) \end{cases}$$

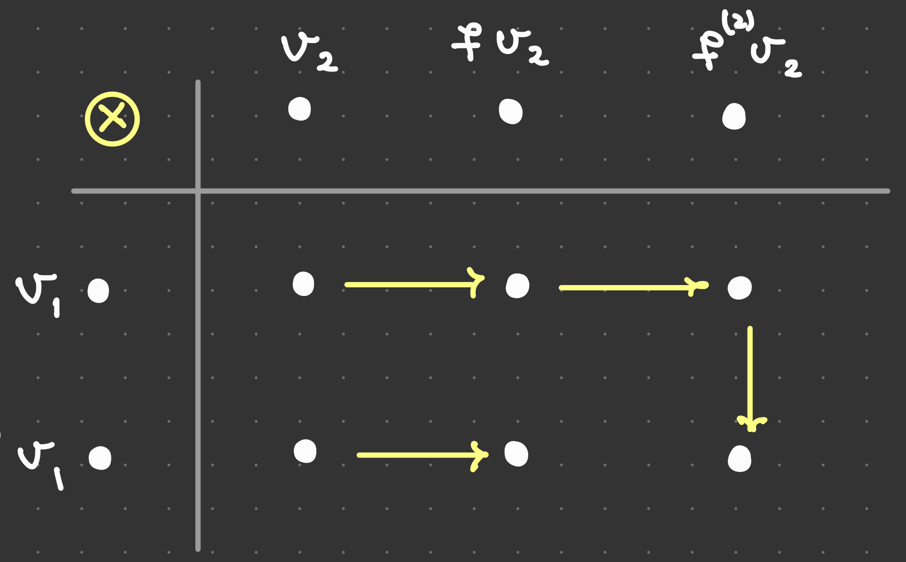
by taking $q = 0$ in the coeff's of $f^{(k)} v_2 \otimes f^{(\ell)} v_1$

$$w_1 = (1+q^2)\, v_2 \otimes f v_1 - q^2 f v_2 \otimes v_1 \equiv v_2 \otimes f v_1$$

$$f w_1 = (1+q^2)\, f v_2 \otimes f v_1 - q^2 [2]\, f^{(2)} v_2 \otimes v_1 - q^2 f v_2 \otimes f v_1$$

$$\equiv f v_2 \otimes f v_1$$

The $U_q(sl_2)$ - strings in $V(2) \otimes V(1)$ at $q = 0$



The above example can be generalized to $V(m) \otimes V(n)$

$$A_0 = \{ f(q) \in K \mid f : \text{regular at } q=0 \} \subset K = \mathbb{k}(q)$$

local ring w/ maximal ideal qA_0

$$V(m) = \bigoplus_{k=0}^{m} f^{(k)} v_m$$

$$L(m) = \bigoplus_{k=0}^{m} A_0 f^{(k)} v_m \quad : \quad A_0\text{-lattice of } V(m)$$

$$B(m) = \{ f^{(k)} v_m \pmod{qL(m)} \mid 0 \leq k \leq m \}$$

$$: \mathbb{k}\text{-basis of } L(m) / qL(m)$$

Note that

$$L(m) \underset{A_0}{\otimes} L(n) \ : \ A_0\text{-lattice of } V(m) \otimes V(n)$$

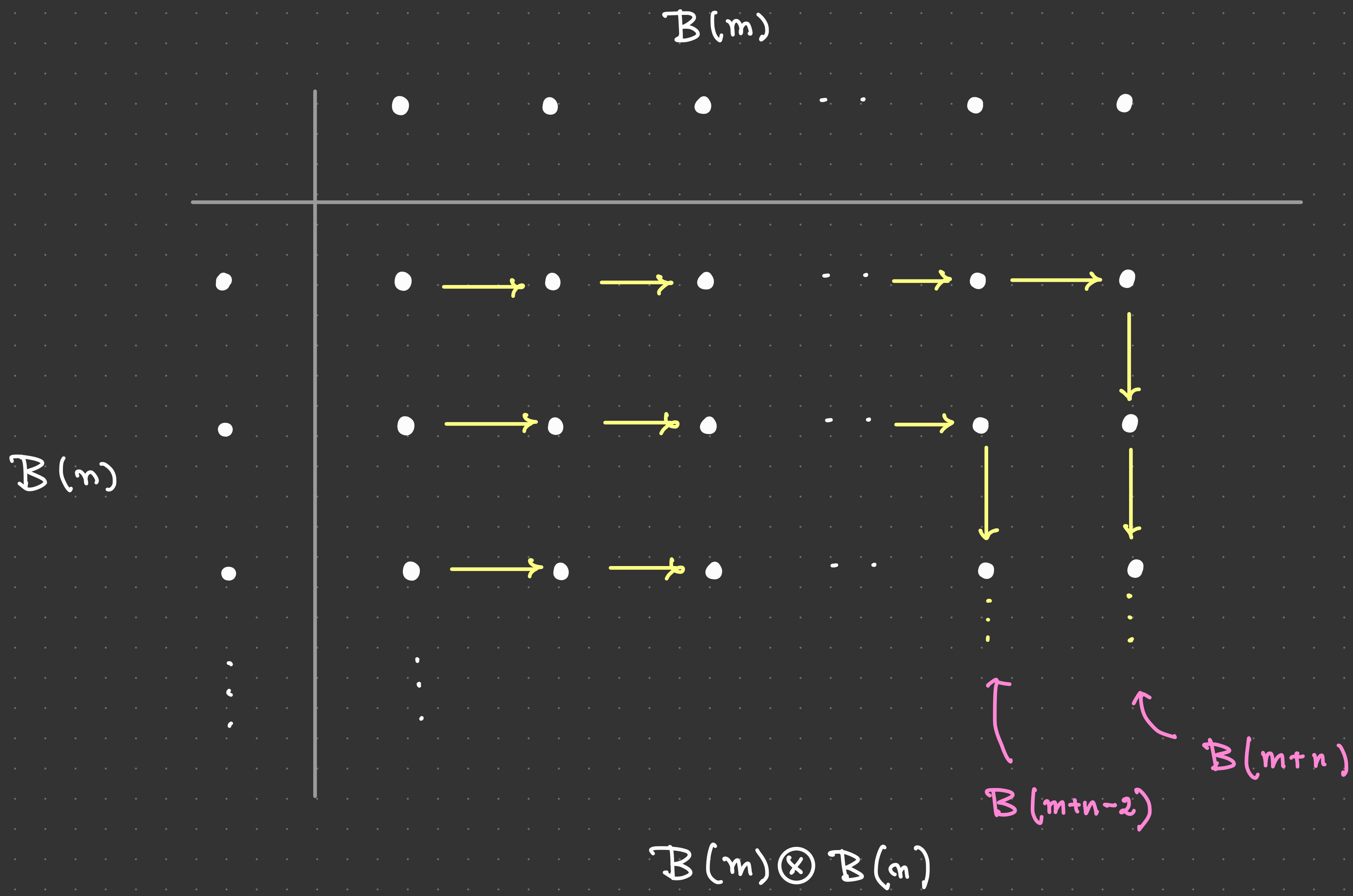
$$B(m) \otimes B(n) \ : \ k\text{-basis of } L(m) \otimes L(n) \big/ q\, L(m) \otimes L(n) \quad \subset \frac{L(m)}{q\,L(m)} \otimes \frac{L(n)}{q\,L(n)}$$

Then we have the following

① $\exists\, w_\ell$: h.w. vector of wt ℓ $\quad (\ell = m+n, \ \cdots, \ \max - \min)$

such that $\quad w_\ell \equiv v_m \otimes f^{(m+n-\ell)} v_n$

② $\quad f^{(k)} w_\ell \equiv f^{(s)} v_m \otimes f^{(t)} v_n \quad$ where (s,t) are given by


B(m)
B(n)
B(m+n-2)
B(m+n)
B(m)⊗B(n)

Rmk $B(m)$

$$\begin{array}{ccccccccccccc}
\bullet & \longrightarrow & \bullet & \longrightarrow & \cdots & \longrightarrow & \bullet & \longrightarrow & \cdots & \longrightarrow & \bullet & \longrightarrow & \bullet \\
v_m & & f v_m & & & & f^{(k)} v_m & & & & f^{(m-1)} v_m & & f^{(m)} v_m \\
\| & & \| & & & & \| & & & & \| & & \| \\
b_0 & & b_1 & & & & b_k & & & & b_{m-1} & & b_m
\end{array}$$

$\varepsilon(b_k) := k$ $\qquad$ $\varphi(b_k) := m - k$

$$\varphi(b_k) - \varepsilon(b_k) = m - 2k = \langle h, wt(b_k) \rangle$$

Then $f^{(k)} w_\ell \pmod{q L(m) \otimes L(n)}$

can be described inductively as follows

If $f^{(k)} w_\ell \equiv \underbrace{f^{(s)} v_m}_{b_1} \otimes \underbrace{f^{(t)} v_n}_{b_2}$, then

$$f^{(k+1)} w_\ell \equiv \begin{cases} f^{(s+1)} v_m \otimes f^{(t)} v_n & \text{if } \varphi(b_1) > \varepsilon(b_2) \\ f^{(s)} v_m \otimes f^{(t+1)} v_n & \text{if } \varphi(b_1) \leq \varepsilon(b_2) \end{cases}$$